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Isomorphism Theorems of Triple Direct Product Convergent Semigroup of Geometric Progression

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Abstract

Given the naturally existing canonical map $\vartheta\left(\left|a^{n+m+l}r^{\frac{n^2}{2}}s^{\frac{m^2}{2}}t^{\frac{l^2}{2}}-z\right|<\varepsilon\right)=\left|(\alpha a)^{n+m+l}(\alpha r)^{\frac{n^2}{2}}(\alpha s)^{\frac{m^2}{2}}(\alpha t)^{\frac{l^2}{2}}-z\right|<\varepsilon$ which is an onto homomorphism, the short-coming of the oneness is eloquently handled by injecting $\ker\vartheta$ into the map as $\vartheta\left(\left(\left|a^{n+m+l}r^{\frac{n^2}{2}}s^{\frac{m^2}{2}}t^{\frac{l^2}{2}}-z\right|<\varepsilon\right)\times ker\vartheta\right)=\left|(\alpha a)^{n+m+l}(\alpha r)^{\frac{n^2}{2}}(\alpha s)^{\frac{m^2}{2}}(\alpha t)^{\frac{l^2}{2}}-z\right|<\varepsilon$ in line with analogous application of the first isomorphism theorem. The second isomorphism theorem is a consequence of $\left(\left|{a_1}^{n+m+l}{r_1}^{\frac{n^2}{2}}{s_1}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)\cup\left(\left|{a_2}^{n+m+l}{r_2}^{\frac{n^2}{2}}{s_2}^{\frac{m^2}{2}}{t_2}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)$ and the third isomorphism theorem gleaned via the definition of the map $\vartheta\left(\left|(\alpha a)^{n+m+l}(\alpha r)^{\frac{n^2}{2}}(\alpha s)^{\frac{m^2}{2}}(\alpha t)^{\frac{l^2}{2}}-z\right|<\varepsilon\right)=\left|(\alpha\beta a)^{n+m+l}(\alpha\beta r)^{\frac{n^2}{2}}(\alpha\beta s)^{\frac{m^2}{2}}(\alpha\beta t)^{\frac{l^2}{2}}-z\right|<\varepsilon$.

Key words: Preston-Wagner Theorem; Oneness; Trichotomy.

MSC2020: 20Mxx (primary), 20Axx (secondary)

1. Introduction

Define $\prod^n u_{a,r}=a^n r^{\frac{n^2}{2}}$, $\prod^m u_{a,s}=a^m s^{\frac{m^2}{2}}$ and $\prod^l u_{a,t}=a^l t^{\frac{l^2}{2}}$. Then $\prod^n u_{a,r}\times\prod^m u_{a,s}\times\prod^l u_{a,t}=a^{n+m+l}r^{\frac{n^2}{2}}s^{\frac{m^2}{2}}t^{\frac{l^2}{2}}$ which converges [1, 2] to z as given $\varepsilon>0$, $\exists\delta>0$ such that $\left|a^{n+m+l}r^{\frac{n^2}{2}}s^{\frac{m^2}{2}}t^{\frac{l^2}{2}}-z\right|<\varepsilon,\forall(n,m,l)\geq N^3\propto\frac{1}{\delta^3}\in\mathbb{N}^3$. Thus, $a^{n+m+l}r^{\frac{n^2}{2}}s^{\frac{m^2}{2}}t^{\frac{l^2}{2}}=\prod u_{a,r,s,t}$. Let $\prod u_{a,r,s,t}=\prod u_{a^4}$, where $a^4=(a,r,s,t)$. Then $a^4\times b^4=(a_1,r_1,s_1,t_1)\times(a_2,r_2,s_2,t_2)=(a_1a_2,r_1r_2,s_1s_2,t_1t_2)=(ab)^4$ and $a^4+b^4=(a+b)^4$. That is, $\prod u_{a^4}\times\prod u_{b^4}=\prod u_{(a\times b)^4}$.

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Using $\prod u_{a_i{}^4} \equiv \prod u_{a_j{}^4}\ mod\ \alpha$ defined by $\prod u_{a_{\frac{i-j}{\alpha}\in\mathbb{Z}}{}^4}$ or $\prod u_{a_{z\in\mathbb{Z}}}$, where $z = \frac{i-j}{\alpha}$, we have equivalence relationship [3, 4]. Thus, define a naturally existing canonical map $\vartheta: \prod u_{a^4} \to {\prod u_{a^4}}/{\prod u_{(\alpha a)^4}}$ by $\vartheta(\prod u_{a^4}) = \prod u_{(\alpha a)^4}$. Then $\vartheta(\prod u_{a^4}) = \prod u_{(\alpha a)^4}$ is an onto homomorphism [5, 6]. For it to be one-one, the $ker\vartheta$ need to be adjoined as follows: $\vartheta: \frac{\prod u_{a^4}}{ker\vartheta} \to \frac{\prod u_{a^4}}{\prod u_{(\alpha a)^4}}$ defined by $\vartheta(\prod u_{a^4} ker\vartheta) = \prod u_{(\alpha a)^4}$ is an isomorphism because: The homomorphism is: $\vartheta[(\prod u_{a_1{}^4} ker\vartheta) \times (\prod u_{a_2{}^4} ker\vartheta)]=\vartheta \quad (\prod u_{(a_2 a_2)^4} ker\vartheta)$ which by definition is $\prod u_{(\alpha a_1 a_2)^4}$. This is $\prod u_{(\alpha a_1)^4} \times \prod u_{(\alpha a_2)^4}$ which is $\vartheta(\prod u_{a_1{}^4} ker\vartheta) \times \vartheta(\prod u_{a_2{}^4} ker\vartheta)$. The oneness is from the fact that $\vartheta(\prod u_{a_1{}^4} ker\vartheta) = \vartheta(\prod u_{a_2{}^4} ker\vartheta)$ implies that $\prod u_{(\alpha a_1)^4} = \prod u_{(\alpha a_2)^4}$. That is, $\prod u_{(a_1)^4} \times \prod u_{(\alpha)^4} = \prod u_{(a_2)^4} \times \prod u_{(\alpha)^4}$ which is $(\prod u_{a_1{}^4} ker\vartheta) = (\prod u_{a_2{}^4} ker\vartheta)$. The ontoness gleaned from $\vartheta^{-1}(\prod u_{(\alpha a_1)^4}) = \vartheta^{-1}(\prod u_{(\alpha a_2)^4}) \implies (\prod u_{a_1{}^4} ker\vartheta) = (\prod u_{a_2{}^4} ker\vartheta)$ which further implies that $\prod u_{(a_1)^4} \times \prod u_{(\alpha)^4} = \prod u_{(a_2)^4} \times \prod u_{(\alpha)^4}$. That is, $\prod u_{(\alpha a_1)^4} = \prod u_{(\alpha a_2)^4}$.

The second isomorphism theorem gleaned from: $\prod u_{(\alpha a_1)^4} \cup \prod u_{(\alpha a_2)^4} = [\prod u_{(\alpha a_1)^4} - \prod u_{(\alpha a_2)^4}] \cup [\prod u_{(\alpha a_1)^4} - \prod u_{(\alpha a_2)^4}] \cup [\prod u_{(\alpha a_1)^4} \cap \prod u_{(\alpha a_2)^4}]$. Let $\prod u_{(\alpha a_1)^4} \Delta \prod u_{(\alpha a_2)^4} = [\prod u_{(\alpha a_1)^4} - \prod u_{(\alpha a_2)^4}] \cup [\prod u_{(\alpha a_1)^4} - \prod u_{(\alpha a_2)^4}]$. Then $\prod u_{(\alpha a_1)^4} \cup \prod u_{(\alpha a_2)^4} = [\prod u_{(\alpha a_1)^4} \Delta \prod u_{(\alpha a_2)^4}] \cup [\prod u_{(\alpha a_1)^4} \cap \prod u_{(\alpha a_2)^4}]$. Dividing both-sides of $\prod u_{(\alpha a_1)^4} \cup \prod u_{(\alpha a_2)^4} = [\prod u_{(\alpha a_1)^4} \Delta \prod u_{(\alpha a_2)^4}] \cup [\prod u_{(\alpha a_1)^4} \cap \prod u_{(\alpha a_2)^4}]$ by, $\prod u_{(\alpha a_2)^4} \cup [\prod u_{(\alpha a_1)^4} \cap \prod u_{(\alpha a_2)^4}]$, we have: $\frac{\prod u_{(\alpha a_1)^4} \cup \prod u_{(\alpha a_2)^4}}{\prod u_{(\alpha a_2)^4} \cup [\prod u_{(\alpha a_1)^4} \cap \prod u_{(\alpha a_2)^4}]}$ is equal to $\frac{[\prod u_{(\alpha a_1)^4} \Delta \prod u_{(\alpha a_2)^4}] \cup [\prod u_{(\alpha a_1)^4} \cap \prod u_{(\alpha a_2)^4}]}{\prod u_{(\alpha a_2)^4} \cup [\prod u_{(\alpha a_1)^4} \cap \prod u_{(\alpha a_2)^4}]}$ and cancelling out common terms, the result follows immediately.

The third isomorphism theorem gleaned from: $\vartheta(\prod u_{(\alpha a)^4}) = \prod u_{(\alpha\beta a)^4}$. The homomorphism is from: $\vartheta(\prod u_{(\alpha a_1)^4} \times \prod u_{(\alpha a_2)^4}) = \vartheta(\prod u_{(\alpha a_1 a_2)^4})$ which by definition is $\prod u_{(\alpha\beta a_1 a_2)^4}$. That is, $\prod u_{(\alpha\beta a_1)^4} \times \prod u_{(\alpha\beta a_2)^4}$. That is, $\vartheta(\prod u_{(\alpha a_1)^4}) \times \vartheta(\prod u_{(\alpha a_2)^4})$. The oneness is from: $\vartheta(\prod u_{(\alpha a_1)^4}) = \vartheta(\prod u_{(\alpha a_2)^4})$ implies $\prod u_{(\alpha\beta a_1)^4} = \prod u_{(\alpha\beta a_2)^4}$. That is $\prod u_{(a_1)^4} \times \prod u_{(\alpha\beta)^4} = \prod u_{(a_2)^4} \times \prod u_{(\alpha\beta)^4}$. That is, $\prod u_{(\alpha a_1)^4} = \prod u_{(\alpha a_2)^4}$ as required.

2. Results

First thing first! We, prove the first, second and third isomorphism theorems chronologically, despite the third isomorphism theorem seems closer to the first than the second.

FIRST ISOMORPHISM THEOREM: Define $\vartheta\left[\left(\left|a^{n+m+l} r^{\frac{n^2}{2}} s^{\frac{m^2}{2}} t^{\frac{l^2}{2}} - z\right| < \varepsilon\right) \times ker\vartheta\right] = \left|(\alpha a)^{n+m+l} (\alpha r)^{\frac{n^2}{2}} (\alpha) s^{\frac{m^2}{2}} (\alpha) t^{\frac{l^2}{2}} - z\right| < \varepsilon$. We show that ϑ is an isomorphism as follows:

Homomorphism: The $\vartheta\left[\left(\left|{a_1}^{n+m+l}{r_1}^{\frac{n^2}{2}}{s_1}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)\times ker\vartheta+\left(\left|{a_2}^{n+m+l}{r_2}^{\frac{n^2}{2}}{s_2}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)\times ker\vartheta\right]$ is $\vartheta\left(\left(\left|(a_1+a_2)^{n+m+l}(r_1+r_2)^{\frac{n^2}{2}}(s_1+s_2)^{\frac{m^2}{2}}(t_1+t_2)^{\frac{l^2}{2}}-z\right|<\varepsilon\right)\times ker\vartheta\right)$ which by definition is $\left|(\alpha(a_1+a_2))^{n+m+l}(\alpha(r_1+r_2))^{\frac{n^2}{2}}(\alpha(s_1+s_2))^{\frac{m^2}{2}}(\alpha(t_1+t_2))^{\frac{l^2}{2}}-z\right|<\varepsilon.$ That is, $\left(\left|(\alpha a_1)^{n+m+l}(\alpha r_1)^{\frac{n^2}{2}}(\alpha s_1)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}}-z\right|<\varepsilon\right)+\left(\left|(\alpha a_2)^{n+m+l}(\alpha r_2)^{\frac{n^2}{2}}(\alpha s_2)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}}-z\right|<\varepsilon\right)$. That is, $\vartheta\left(\left(\left|{a_1}^{n+m+l}{r_1}^{\frac{n^2}{2}}{s_1}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)\times ker\vartheta\right)+\vartheta\left(\left(\left|{a_2}^{n+m+l}{r_2}^{\frac{n^2}{2}}{s_2}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)\times ker\vartheta\right)$.

Oneness: Let $\vartheta\left(\left(\left|{a_1}^{n+m+l}{r_1}^{\frac{n^2}{2}}{s_1}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)\times ker\vartheta\right)=\vartheta\left(\left(\left|{a_2}^{n+m+l}{r_2}^{\frac{n^2}{2}}{s_2}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)\times ker\vartheta\right)$. Then $\left(\left|(\alpha a_1)^{n+m+l}(\alpha r_1)^{\frac{n^2}{2}}(\alpha s_1)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}}-z\right|<\varepsilon\right)=\left(\left|(\alpha a_2)^{n+m+l}(\alpha r_2)^{\frac{n^2}{2}}(\alpha s_2)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}}-z\right|<\varepsilon\right)$. That is, $\left(\left|{a_1}^{n+m+l}{r_1}^{\frac{n^2}{2}}{s_1}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\times(\left|\alpha^{n+m+l}\alpha^{\frac{n^2}{2}}\alpha^{\frac{m^2}{2}}\alpha^{\frac{l^2}{2}}-z\right|<\varepsilon)\right)+\left(\left|{a_1}^{n+m+l}{r_1}^{\frac{n^2}{2}}{s_1}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\times(\left|\alpha^{n+m+l}\alpha^{\frac{n^2}{2}}\alpha^{\frac{m^2}{2}}\alpha^{\frac{l^2}{2}}-z\right|<\varepsilon)\right)+\cdots\ \alpha\ times$ is equal to $\left(\left|{a_2}^{n+m+l}{r_2}^{\frac{n^2}{2}}{s_2}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\times(\left|\alpha^{n+m+l}\alpha^{\frac{n^2}{2}}\alpha^{\frac{m^2}{2}}\alpha^{\frac{l^2}{2}}-z\right|<\varepsilon)\right)+\left(\left|{a_2}^{n+m+l}{r_2}^{\frac{n^2}{2}}{s_2}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\times(\left|\alpha^{n+m+l}\alpha^{\frac{n^2}{2}}\alpha^{\frac{m^2}{2}}\alpha^{\frac{l^2}{2}}-z\right|<\varepsilon)\right)+\cdots\alpha\ times.$ That is, $\left(\left|{a_1}^{n+m+l}{r_1}^{\frac{n^2}{2}}{s_1}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)\times ker\vartheta=\left(\left|{a_2}^{n+m+l}{r_2}^{\frac{n^2}{2}}{s_2}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)\times ker\vartheta.$

Ontoness: Let $\vartheta^{-1}\left(\left|(\alpha a_1)^{n+m+l}(\alpha r_1)^{\frac{n^2}{2}}(\alpha s_1)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}}-z\right|<\varepsilon\right)=\vartheta^{-1}\left(\left|(\alpha a_2)^{n+m+l}(\alpha r_2)^{\frac{n^2}{2}}(\alpha s_2)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}}-z\right|<\varepsilon\right)$. Then, by definition, it is: $\left|{a_1}^{n+m+l}{r_1}^{\frac{n^2}{2}}{s_1}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\times(ker\vartheta)=\left|{a_2}^{n+m+l}{r_2}^{\frac{n^2}{2}}{s_2}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\times(ker\vartheta)$. That is, $\left(\left|{a_1}^{n+m+l}{r_1}^{\frac{n^2}{2}}{s_1}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\times(ker\vartheta)\right)+\left(\left|{a_1}^{n+m+l}{r_1}^{\frac{n^2}{2}}{s_1}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\times(ker\vartheta)\right)+\cdots\ \alpha\ times$ is equal to $\left(\left|{a_2}^{n+m+l}{r_2}^{\frac{n^2}{2}}{s_2}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\times(ker\vartheta)\right)+\left(\left|{a_2}^{n+m+l}{r_2}^{\frac{n^2}{2}}{s_2}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\times(ker\vartheta)\right)+\cdots\alpha\ times.$ That is, $\left|(\alpha a_1)^{n+m+l}(\alpha r_1)^{\frac{n^2}{2}}(\alpha s_1)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}}-z\right|<\varepsilon=\left|(\alpha a_2)^{n+m+l}(\alpha r_2)^{\frac{n^2}{2}}(\alpha s_2)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}}-z\right|<\varepsilon.$

THE SECOND ISOMORPHISM THEOREM: The $\left(\left|{a_1}^{n+m+l}{r_1}^{\frac{n^2}{2}}{s_1}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)\cup\left(\left|{a_2}^{n+m+l}{r_2}^{\frac{n^2}{2}}{s_2}^{\frac{m^2}{2}}{t_2}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)$ is equal to: $\left[\left(\left|{a_1}^{n+m+l}{r_1}^{\frac{n^2}{2}}{s_1}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)-\left(\left|{a_2}^{n+m+l}{r_2}^{\frac{n^2}{2}}{s_2}^{\frac{m^2}{2}}{t_2}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)\right]\cup\left[\left(\left|{a_2}^{n+m+l}{r_2}^{\frac{n^2}{2}}{s_2}^{\frac{m^2}{2}}{t_2}^{\frac{l^2}{2}}-z\right|<\varepsilon\right)-\left(\left|{a_1}^{n+m+l}{r_1}^{\frac{n^2}{2}}{s_1}^{\frac{m^2}{2}}{t_1}^{\frac{l^2}{2}}-\right.\right.\right.$

$z\Big| < \varepsilon\Big)\Big] \cup \Big[\Big(\Big|a_1{}^{n+m+l}r_1{}^{\frac{n^2}{2}}s_1{}^{\frac{m^2}{2}}t_1{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) \cap \Big(\Big|a_2{}^{n+m+l}r_2{}^{\frac{n^2}{2}}s_2{}^{\frac{m^2}{2}}t_2{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)\Big]$. Let $\Big[\Big(\Big|a_1{}^{n+m+l}r_1{}^{\frac{n^2}{2}}s_1{}^{\frac{m^2}{2}}t_1{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) \Delta \Big(\Big|a_2{}^{n+m+l}r_2{}^{\frac{n^2}{2}}s_2{}^{\frac{m^2}{2}}t_2{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)\Big] = \Big[\Big(\Big|a_2{}^{n+m+l}r_2{}^{\frac{n^2}{2}}s_2{}^{\frac{m^2}{2}}t_2{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) - \Big(\Big|a_1{}^{n+m+l}r_1{}^{\frac{n^2}{2}}s_1{}^{\frac{m^2}{2}}t_1{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)\Big] \cup \Big[\Big(\Big|a_1{}^{n+m+l}r_1{}^{\frac{n^2}{2}}s_1{}^{\frac{m^2}{2}}t_1{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) - \Big(\Big|a_2{}^{n+m+l}r_2{}^{\frac{n^2}{2}}s_2{}^{\frac{m^2}{2}}t_2{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)\Big]$. Then $\Big[\Big(\Big|a_1{}^{n+m+l}r_1{}^{\frac{n^2}{2}}s_1{}^{\frac{m^2}{2}}t_1{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) \cup \Big(\Big|a_2{}^{n+m+l}r_2{}^{\frac{n^2}{2}}s_2{}^{\frac{m^2}{2}}t_2{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)\Big] = \Big[\Big(\Big|a_2{}^{n+m+l}r_2{}^{\frac{n^2}{2}}s_2{}^{\frac{m^2}{2}}t_2{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) \Delta \Big(\Big|a_1{}^{n+m+l}r_1{}^{\frac{n^2}{2}}s_1{}^{\frac{m^2}{2}}t_1{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)\Big] \cup \Big[\Big(\Big|a_1{}^{n+m+l}r_1{}^{\frac{n^2}{2}}s_1{}^{\frac{m^2}{2}}t_1{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) \cap \Big(\Big|a_2{}^{n+m+l}r_2{}^{\frac{n^2}{2}}s_2{}^{\frac{m^2}{2}}t_2{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)\Big]$. Dividing both-sides by $\Big(\Big|a_2{}^{n+m+l}r_2{}^{\frac{n^2}{2}}s_2{}^{\frac{m^2}{2}}t_2{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) \cup \Big[\Big(\Big|a_1{}^{n+m+l}r_1{}^{\frac{n^2}{2}}s_1{}^{\frac{m^2}{2}}t_1{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) \cap \Big(\Big|a_2{}^{n+m+l}r_2{}^{\frac{n^2}{2}}s_2{}^{\frac{m^2}{2}}t_2{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)\Big]$ and cancelling out common terms, we have:

$$\frac{\Big(\Big|a_1{}^{n+m+l}r_1{}^{\frac{n^2}{2}}s_1{}^{\frac{m^2}{2}}t_1{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)}{\Big(\Big|a_1{}^{n+m+l}r_1{}^{\frac{n^2}{2}}s_1{}^{\frac{m^2}{2}}t_1{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) \cap \Big(\Big|a_2{}^{n+m+l}r_2{}^{\frac{n^2}{2}}s_2{}^{\frac{m^2}{2}}t_2{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)} = \frac{\Big(\Big|a_2{}^{n+m+l}r_2{}^{\frac{n^2}{2}}s_2{}^{\frac{m^2}{2}}t_2{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) \Delta \Big(\Big|a_1{}^{n+m+l}r_1{}^{\frac{n^2}{2}}s_1{}^{\frac{m^2}{2}}t_1{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)}{\Big(\Big|a_2{}^{n+m+l}r_2{}^{\frac{n^2}{2}}s_2{}^{\frac{m^2}{2}}t_2{}^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)}.$$

THIRD ISOMORPHISM THEOREM: Define $\vartheta\Big((\alpha a)^{n+m+l}(\alpha r)^{\frac{n^2}{2}}(\alpha)s^{\frac{m^2}{2}}(\alpha)t^{\frac{l^2}{2}} - z\Big) = \Big|(\alpha\beta a)^{n+m+l}(\alpha\beta r)^{\frac{n^2}{2}}(\alpha\beta)s^{\frac{m^2}{2}}(\alpha\beta)t^{\frac{l^2}{2}} - z\Big| < \varepsilon$. We show that ϑ is an onto homomorphism as follows:

Homomorphism: The $\vartheta\Big[\Big(\Big|(\alpha a_1)^{n+m+l}(\alpha r_1)^{\frac{n^2}{2}}(\alpha s_1)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) + \Big(\Big|(\alpha a_2)^{n+m+l}(\alpha r_2)^{\frac{n^2}{2}}(\alpha s_2)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)\Big]$ is $\vartheta\Big(\Big|((\alpha a_1) + (\alpha a_2))^{n+m+l}((\alpha r_1) + (\alpha r_2))^{\frac{n^2}{2}}((\alpha s_1) + ((\alpha s_2))^{\frac{m^2}{2}}((\alpha t_1) + (\alpha t_2))^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)$ which by definition is $\Big|(\alpha\beta(a_1 + a_2))^{n+m+l}(\alpha\beta(r_1 + r_2))^{\frac{n^2}{2}}(\alpha\beta(s_1 + s_2))^{\frac{m^2}{2}}(\alpha\beta(t_1 + t_2))^{\frac{l^2}{2}} - z\Big| < \varepsilon$. That is, $\Big(\Big|(\alpha\beta a_1)^{n+m+l}(\alpha\beta r_1)^{\frac{n^2}{2}}(\alpha\beta s_1)^{\frac{m^2}{2}}(\alpha\beta t_1)^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) + \Big(\Big|(\alpha\beta a_2)^{n+m+l}(\alpha\beta r_2)^{\frac{n^2}{2}}(\alpha\beta s_2)^{\frac{m^2}{2}}(\alpha\beta t_1)^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)$. That is, $\vartheta\Big(\Big|(\alpha a_1)^{n+m+l}(\alpha r_1)^{\frac{n^2}{2}}(\alpha s_1)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) + \vartheta\Big(\Big|(\alpha a_2)^{n+m+l}(\alpha r_2)^{\frac{n^2}{2}}(\alpha s_2)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)$.

Oneness: Let $\vartheta\Big(\Big|(\alpha a_1)^{n+m+l}(\alpha r_1)^{\frac{n^2}{2}}(\alpha s_1)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) = \vartheta\Big(\Big|(\alpha a_2)^{n+m+l}(\alpha r_2)^{\frac{n^2}{2}}(\alpha s_2)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)$. Then, by definition; $\Big(\Big|(\alpha\beta a_1)^{n+m+l}(\alpha\beta r_1)^{\frac{n^2}{2}}(\alpha\beta s_1)^{\frac{m^2}{2}}(\alpha\beta t_1)^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big) + \Big(\Big|(\alpha\beta a_2)^{n+m+l}(\alpha\beta r_2)^{\frac{n^2}{2}}(\alpha\beta s_2)^{\frac{m^2}{2}}(\alpha\beta t_1)^{\frac{l^2}{2}} - z\Big| < \varepsilon\Big)$.

That is, $\left(\left|(\alpha a_1)^{n+m+l}(\alpha r_1)^{\frac{n^2}{2}}(\alpha s_1)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon\right) + \left(\left|(\alpha a_1)^{n+m+l}(\alpha r_1)^{\frac{n^2}{2}}(\alpha s_1)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon\right) + \cdots \beta\ times$ is equal to $\left(\left|(\alpha a_2)^{n+m+l}(\alpha r_2)^{\frac{n^2}{2}}(\alpha s_2)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon\right) + \left(\left|(\alpha a_2)^{n+m+l}(\alpha r_2)^{\frac{n^2}{2}}(\alpha s_2)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon\right) + \cdots \beta\ times$. That is, $\left|(\alpha a_1)^{n+m+l}(\alpha r_1)^{\frac{n^2}{2}}(\alpha s_1)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon = \left|(\alpha a_2)^{n+m+l}(\alpha r_2)^{\frac{n^2}{2}}(\alpha s_2)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon$.

Ontoness: Let $\vartheta^{-1}\left(\left|(\alpha\beta a_1)^{n+m+l}(\alpha\beta r_1)^{\frac{n^2}{2}}(\alpha\beta s_1)^{\frac{m^2}{2}}(\alpha\beta t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon\right) = \vartheta^{-1}\left(\left|(\alpha\beta a_2)^{n+m+l}(\alpha\beta r_2)^{\frac{n^2}{2}}(\alpha\beta s_2)^{\frac{m^2}{2}}(\alpha\beta t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon\right)$. Then, by definition, it is: $\left(\left|(\alpha a_1)^{n+m+l}(\alpha r_1)^{\frac{n^2}{2}}(\alpha s_1)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon\right) = \left(\left|(\alpha a_2)^{n+m+l}(\alpha r_2)^{\frac{n^2}{2}}(\alpha s_2)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon\right)$. That is, $\left(\left|(\alpha a_1)^{n+m+l}(\alpha r_1)^{\frac{n^2}{2}}(\alpha s_1)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon\right) + \left(\left|(\alpha a_1)^{n+m+l}(\alpha r_1)^{\frac{n^2}{2}}(\alpha s_1)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon\right) + \cdots \beta\ times$ is equal to $\left(\left|(\alpha a_2)^{n+m+l}(\alpha r_2)^{\frac{n^2}{2}}(\alpha s_2)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon\right) + \left(\left|(\alpha a_2)^{n+m+l}(\alpha r_2)^{\frac{n^2}{2}}(\alpha s_2)^{\frac{m^2}{2}}(\alpha t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon\right) + \cdots \beta\ times$. That is, $\left(\left|(\alpha\beta a_1)^{n+m+l}(\alpha\beta r_1)^{\frac{n^2}{2}}(\alpha\beta s_1)^{\frac{m^2}{2}}(\alpha\beta t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon\right) = \left(\left|(\alpha\beta a_2)^{n+m+l}(\alpha\beta r_2)^{\frac{n^2}{2}}(\alpha\beta s_2)^{\frac{m^2}{2}}(\alpha\beta t_1)^{\frac{l^2}{2}} - z\right| < \varepsilon\right)$.

3. Conclusion

The isomorphism theorems were first invented by [7] few years before the birth of both group [8] and semigroup [9]. However, the first isomorphism theorem [10] overcome the deficiency of oneness of the naturally existing canonical map $\vartheta: S \to {}^{S}/_{I}$ by intruding kernel of homomorphism, $\ker \vartheta$, where ${}^{S}/_{I} = Im\vartheta$. This makes $\vartheta(s) = Is$ to become $\vartheta(Is) = Is$ which is an isomorphism. The second isomorphism [11] gleaned from $S \cup I$ and the third isomorphism is a recursive use of the first isomorphim theorem as $\vartheta(Is\alpha) = Is\alpha\beta$ a route to composition series of solvability as in for example [12].

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