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Q-derivative of Modifie Tremblay Operator

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Abstract

A new class of analytic function defined by q-derivative of modified Tremblay fractional derivative operator in the unit disk was established. Coefficient bounds for $|a_2|$, $|a_3|$ and $|a_4|$ were obtained. Furthermore, Fekete - Szego estimate of functions belonging to the class $S^{q}{}_{v,l,\beta,b}(z)$ was derived. The results obtained generalized some earlier ones in literature.

Keywords: Univalent function; Coefficient bounds; Tremblay operator.

1. Introduction

Let C, R and N denote the set of all complex numbers, real numbers and positive integers. The above shall be employed throughout the present investigation. Let $\chi(n)$ denote the class of functions $f(z)$ which are normalized with the conditions

$$f(0) = f'(0) - 1 = 0$$

in the form

$$f(z) = z + c_{n+1}z^{n+1} + c_{n+2}z^{n+2} + \ldots, \qquad n \in N_1 \tag{1.0}$$

and analytic in the open unit disk $U = \{z \in C : |z| \le 1\}$.

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Recently, some researchers developed interest in fractional derivative operator, particularly of order $u(0 \le u, < 1)$. This is likely due to its application in various field of endeavor.

1. Preliminaries

The well-known fractional derivative operator is given as

$$D_z^v k[z], 0 \le v < 1$$

And it is presented integrally as

$$D_z^v k[z] = \frac{1}{\Gamma(1-v)} \int_0^z \frac{k(\varepsilon)}{(z-\varepsilon)^v} d\varepsilon, 0 \le v < 1 \qquad (2.1)$$

Where k(z) is an analytic function in a simply-connected complex domain containing the origin. The multiplicity of $(z-\varepsilon)^v$ can be removed by introducing $\log(z-\varepsilon)$ which is real whenever $z-\varepsilon > 0$. The Γ is the popular gamma function. Let $k \in N_0$ and 0 ≤ υ < 1; using (2.1) we obtain the Srivastava-Owa derivation of order $k+v$ given as

$$D_z^{k+v} k[z] = \frac{dz}{dz^i} D_z^{k+v} k[z]$$

or

$$D_z^{0+v} k[z] = D_z^v k[z] \, and \, D_z^{1+v} k[z] = \frac{d}{dz} D_z^v k[z]$$

Suppose f(z) ∈ χ(n) and of the form (1.1), the Tremblay operator denoted by $T_{i,v}[f](z), T_{i,v}[f(z)] \, or \, T_{i,v}[f]$ is defined as

$$T_{i,v}[f(z)] = \frac{\Gamma(v)}{\Gamma(i)} z^{1-v} D^{i-v} z^{i-1}[f(z)] \qquad (2.2)$$

For $0 < i < 1, 0 < v \le 1, 0 \le i - v < 1$ and z ∈ U, the operator $D_z^{i-v}[z^{i-1} f(z)]$ coincides with that of Srivastava-Owa operator earlier mentioned. From (2.2) the following results are obtainable:

$$T_{i,v}[f(z)] = \frac{i}{v} z + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+i)\Gamma(v)}{\Gamma(k+v)\Gamma(i)} a_k z^k \qquad (2.3)$$

And

$$D_q\left(T_{i,v}\left[f(z)\right]\right) = \frac{i}{v}z + \sum_{k=n+1}^{\infty} \frac{[k]_q \Gamma(k+i)\Gamma(v)}{\Gamma(k+v)\Gamma(i)} a_k z^{k-1} \tag{2.4}$$

It is interesting to note that from all the available information due to (2.1)-(2.4) and taking cognizance of admissible values related to υ and ι then the following hold

$$D_q\left(T_{i,v}\left[f(z)\right]\right) \in \chi_n, \tag{2.5}$$

$$D_q\left(T_{1,1}\left[f(z)\right]\right) = f_0(z) \in \chi_n, \tag{2.6}$$

$$D_q\left(T_{1,1}\left[f(z)\right]\right) = f(z) \in \chi_n z \in U, \tag{2.7}$$

In 2016, Esa et.al.[4] modified the Tremblay operator and defined as :

$$T_z^{v,i} f(z) = \frac{i}{v} T_z^{v,i} f(z)$$
$$T_z^{v,i} f(z) = \frac{\Gamma(i+1)}{\Gamma(v+1)} z^{1-i} D_z^{\,v-1} f(z)$$

Simply we have

$$T_z^{v,i} f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+i)\Gamma(i+1)}{\Gamma(k+v)\Gamma(v+1)} a_k z^k \tag{2.8}$$

This work was motivated by Irmark and Olga [10] and Esa et.al [4].

Definition 2.1 Let $\varphi(z) \in P$, $0 \le i \le 1, 0 \le v \le 1, 0 < q < 1, \beta \ge 0, s \ge 0, b \in C \setminus \{0\}$. A function $T_z^{v,i} f(z) \in A$ of the form (2.8) is said to belong to the class $S^q{}_{v,t,\beta,b}(z)$ if

$$1 + \frac{1}{b}\left((1 + i\tan\beta) \frac{zD_q T_{i,v} f(z)}{T_{i,v} f(z)} - i\tan\beta - 1 \right) < \varphi(z)$$

Remark 2.1 The following are some classes generalized by the class $S^q{}_{v,t,\beta,b}(z)$. Let q → 1,

1. for $\varphi(z) = \frac{1+A_z}{1+B_z}, \beta = 0, b = 1$ the class reduced to $S^{\beta}(A,B)$ introduced and investigated by [16]

2. if $\beta = 0, i = v = 1$ the class reduce to starlike function

3. if $\beta = 0, i = v = 1$ the class reduce to class $S_b^*(\varphi)$ introduced and studied by [12]

The useful lemma for this work is stated below:

Lemma 2.1 If $\omega \in \Omega$ then

$$\left|\omega_2 - t\omega_1^2\right| \le \max 1, |t|$$

for any complex number t. The result is sharp for the function $\omega z = z \ or \ \omega(z) z_2$, (see also [9])

Lemma 2.2(Schwarz lemma)

Let the analytic function $\varphi(z)$ be regular in U and let $\varphi(0) = 0$ .If $|\varphi(z)| \le 1$ in the open unit disk U, then

$$|\varphi(z)| \le |z| \ (|z| < 1) \ \text{ and } \ |\varphi'(0)| \le 1$$

Main Results

Theorem 3.1 Let $\varphi(z) = 1 + B_1 z + B_2 z^3 + B_3 z^3 + \ldots$ if $T_{i,v} f(z)$ given by (2.8) belongs the class $S^q{}_{v,t,\beta,b}(z)$ then,

$$|a_2| \le \frac{|b| B_1 d_1 \Gamma(2+v)\Gamma(i+1)}{(1 + i\tan\beta)([2]_q - 1)\Gamma(2+i)\Gamma(v+1)}$$

$$|a_3| \le \frac{2|b| B_1 d_1 \Gamma(3+v)\Gamma(i+1)}{2(1 + i\tan\beta)([3]_q - 1)\Gamma(3+i)\Gamma(v+1)} \left(d_2 + \left(\frac{bB_1}{(1 + i\tan\beta)([2]_q - 1)} + \frac{bB_2}{4B_1} - \frac{1}{2} \right) d_1^2 \right)$$

Proof:

Suppose $T_{i,v} f(z) \in S^q{}_{v,t,\beta,b}(z)$ by definition

$$1 + \frac{1}{b}\left((1 + i\tan\beta) \frac{z\partial_q T_{i,v} f(z)}{T_{i,v} f(z)} - i\tan\beta - 1 \right) \prec \varphi(z)$$

Therefore,

$$1+\frac{1}{b}\left((1+i\tan\beta)\frac{z\partial_q T_{i,v} f(z)}{T_{i,v} f(z)}-i\tan\beta-1\right)=\varphi(\omega(z)) \tag{3.1}$$

Now,

$$\frac{zD_q T_{i,v} f(z)}{T_{i,v} f(z)}=\frac{z+\sum_{k=2}^{\infty}[k]_q \frac{\Gamma(k+i)\Gamma(v+1)}{\Gamma(v+i)\Gamma(i+1)} a_k z^k}{z+\sum_{k=2}^{\infty}[k]_q \frac{\Gamma(k+i)\Gamma(v+1)}{\Gamma(v+i)\Gamma(i+1)} a_k z^k}=\frac{1+\sum_{k=2}^{\infty}[k]_q \frac{\Gamma(k+i)\Gamma(v+1)}{\Gamma(v+i)\Gamma(i+1)} a_k z^{k-1}}{1+\sum_{k=2}^{\infty}[k]_q \frac{\Gamma(k+i)\Gamma(v+1)}{\Gamma(v+i)\Gamma(i+1)} a_k z^{k-1}}$$

$$\begin{aligned}1+\frac{1}{b}\left((1+i\tan\beta)\frac{z\partial_q T_{i,v} f(z)}{T_{i,v} f(z)}-i\tan\beta-1\right)&=1+\frac{(1+i\tan\beta)}{b}\left([2]_q-1\right)\frac{\Gamma(2+i)^2\Gamma(v+1)^2}{\Gamma(2+v)^2\Gamma(i+1)^2}a_2 z\\&+\frac{(1+i\tan\beta)}{b}\left(\left([3]_q-1\right)\frac{\Gamma(3+i)\Gamma(v+1)}{\Gamma(3+v)\Gamma(i+1)}a_3-\left([2]_q-1\right)\frac{\Gamma(2+i)^2\Gamma(v+1)^2}{\Gamma(2+v)^2\Gamma(i+1)^2}{a_2}^2\right)z^2\\&+\frac{(1+i\tan\beta)}{b}\left(\left([4]_q-1\right)\frac{\Gamma(4+i)\Gamma(v+1)}{\Gamma(4+v)\Gamma(i+1)}a_4-\left([3]_q-[2]_q\right)\frac{\Gamma(2+i)\Gamma(3+i)\Gamma(v+1)^2}{\Gamma(2+v)\Gamma(3+v)\Gamma(i+1)^2}a_2 a_3\right)z^3+\ldots\end{aligned}$$

Let,

$$P(z)=\frac{1+\omega(z)}{1-\omega(z)}=\frac{1}{2}\left(d_1 z+\left(d_2-\frac{{d_1}^2}{2}\right)z^2+\ldots\right) \tag{3.2}$$

Now,

$$\phi(\omega(z))=1+\frac{B_1 d_1}{2}z+\left(\frac{B_1 d_2}{2}+\frac{(B_2-B_1){d_1}^2}{4}\right)z^2+\left(\frac{B_1 d_3}{2}+\frac{(B_2-B_1)d_1 d_2}{2}+\frac{\left(B_1-2B_2+B_3{d_1}^3\right)}{8}\right)z^3+\ldots$$

Since,

$$1+\frac{1}{b}\left((1+i\tan\beta)\frac{zD_q T_{i,v} f(z)}{T_{i,v} f(z)}-i\tan\beta-1\right)=\varphi(\omega(z))$$

We have

$$\begin{aligned}&1+\frac{(1+i\tan\beta)}{b}\left([2]_q-1\right)\frac{\Gamma(2+i)^2\Gamma(v+1)^2}{\Gamma(2+v)^2\Gamma(i+1)^2}a_2 z\\&+\frac{(1+i\tan\beta)}{b}\left(\left([3]_q-1\right)\frac{\Gamma(3+i)\Gamma(v+1)}{\Gamma(3+v)\Gamma(i+1)}a_3-\left([2]_q-1\right)\frac{\Gamma(2+i)^2\Gamma(v+1)^2}{\Gamma(2+v)^2\Gamma(i+1)^2}{a_2}^2\right)z^2\end{aligned}$$

$$+\frac{(1+i\tan\beta)}{b}\left(([4]_q-1)\frac{\Gamma(4+i)\Gamma(v+1)}{\Gamma(4+v)\Gamma(i+1)}a_4-([3]_q-[2]_q)\frac{\Gamma(2+i)\Gamma(3+i)\Gamma(v+1)^2}{\Gamma(2+v)\Gamma(3+v)\Gamma(i+1)^2}a_2a_3\right)z^3+\dots$$
$$=1+\frac{B_1d_1}{2}z+\left(\frac{B_1d_2}{2}+\frac{(B_2-B_1)d_1^{\,2}}{4}\right)z^2+\left(\frac{B_1d_3}{2}+\frac{(B_2-B_1)d_1d_2}{2}+\frac{(B_1-2B_2+B_3d_1^{\,3})}{8}\right)z^3+\dots$$

Comparing coefficients of z, z^2, z^3, we have

$$|a_2|\le\frac{|b|B_1d_1\Gamma(2+v)\Gamma(i+1)}{(1+i\tan\beta)([2]_q-1)\Gamma(2+i)\Gamma(v+1)} \tag{3.3}$$

$$|a_3|\le\frac{2|b|B_1d_1\Gamma(3+v)\Gamma(i+1)}{2(1+i\tan\beta)([3]_q-1)\Gamma(3+i)\Gamma(v+1)}\left(d_2+\left(\frac{bB_1}{(1+i\tan\beta)([2]_q-1)}+\frac{bB_2}{4B_1}-\frac{1}{2}\right)d_1^{\,2}\right) \tag{3.4}$$

Which completes the proof.

Corollary 3.1 Suppose $T_{i,v}f(z)$ belongs to the class $S^q{}_{1,1,0\beta,b}(z)$ we obtained

$$|a_2|\le\frac{|b|B_1d_1}{(1+i\tan\beta)}\text{ and}$$

$$|a_3|\le\frac{2B_1}{2(1+i\tan\beta)}\left(d_2+\left(\frac{bB_1}{(1+i\tan\beta)}+\frac{bB_2}{4B_1}-\frac{1}{2}\right)d_1^{\,2}\right)$$

Remark 3.1 This is the result obtained in [12]

Theorem 3.2 Let $T_{i,v}f(z)$ be the function given by (2.8) and belongs to the class $S^q{}_{v,t,\beta,b}(z)$ then for any complex number $v\in C$

$$\left|a_3-va_2^{\,2}\right|\le\frac{2|b|B_1\Gamma(3+v)\Gamma(i+1)}{2(1+i\tan\beta)([3]_q-1)\Gamma(3+i)\Gamma(v+1)}\max\{1,v\}$$
$$where,$$
$$v=\left(\frac{bB_1}{(1+i\tan\beta)}+\frac{bB_2}{4B_1}-\frac{1}{2}+\frac{2bB_1\Gamma(3+i)\Gamma(v+1)\Gamma(2+v)^2\Gamma(1+i)^2}{(1+i\tan\beta)([3]_q-1)\Gamma(3+v)^2\Gamma(i+1)\Gamma(2+i)^2\Gamma(1+v)^2}\right)$$

Proof:

Let $T_{i,v} f(z) \in S^{q}{}_{v,t,\beta,b}(z)$ then substituting (3.3) and (3.4) into $\left|a_3 - ka_2{}^2\right|$ we have

$$a_3 - va_2{}^2 \le \frac{2bB_1\Gamma(3+v)\Gamma(i+1)}{2(1+i\tan\beta)([3]_q - 1)\Gamma(3+i)\Gamma(v+1)}\left(d_2\left(\frac{bB_1}{2(1+i\tan\beta)([2]_q - 1)} + \frac{bB_2}{4B_1} - \frac{1}{2}\right)d_1{}^2\right)$$

$$- v\frac{b^2 B_1{}^2 d_1{}^2 \Gamma(2+v)^2 \Gamma(i+1)^2}{(1+i\tan\beta)^2([2]_q - 1)^2\Gamma(2+i)^2\Gamma(v+1)^2}$$

then,

$$\left|a_3 - va_2{}^2\right| = \left|\frac{2bB_1\Gamma(3+v)\Gamma(i+1)}{2(1+i\tan\beta)([3]_q - 1)\Gamma(3+i)\Gamma(v+1)}\left(d_2 - vd_1{}^2\right)\right|$$

$$\left|a_3 - va_2{}^2\right| \le \frac{2|b|B_1\Gamma(3+v)\Gamma(i+1)}{2(1+i\tan\beta)([3]_q - 1)\Gamma(3+i)\Gamma(v+1)}\left|d_2 - vd_1{}^2\right| \qquad (3.6)$$

Using lemma (2.1) in equation (3.6), we have

$$\left|a_3 - va_2{}^2\right| \le \frac{2|b|B_1\Gamma(3+v)\Gamma(i+1)}{2(1+i\tan\beta)([3]_q - 1)\Gamma(3+i)\Gamma(v+1)}\max\{1,|v|\}$$

where

$$v = v\left(\frac{bB_1}{2(1+i\tan\beta)([2]_q - 1)} + \frac{bB_2}{4B_1} - \frac{1}{2} + \frac{2bB_1\Gamma(3+i)\Gamma(v+1)\Gamma(2+v)^2\Gamma(1+i)^2}{(1+i\tan\beta)([3]_q - 1)\Gamma(3+v)^2\Gamma(i+1)\Gamma(2+i)^2\Gamma(1+v)^2}\right)$$

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