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Reproducing Kernel for the Classical Dirichlet Space of the Upper Half Plane

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Abstract

We determine the reproducing kernel for the classical Dirichlet space of the upper half plane, $\mathcal{D}(\mathbb{U})$. Consequently, we establish the norm of the reproducing kernel and growth condition for functions in $\mathcal{D}(\mathbb{U})$. Moreover, we extend the existing relationship between the reproducing kernels for the classical Dirichlet space and Bergman space of the unit disk to their counterparts of the upper half plane.

Keywords: Classical Dirichlet Space; Reproducing kernel; Norm; Growth condition.

1. Introduction

Let $\mathbb{C}$ denote the complex plane. On $\Omega \subseteq \mathbb{C}$, an open set, we define a Hilbert space of functions, denoted *H*. Then, $K : \Omega \times \Omega \to \mathbb{C}$ is deemed to be a reproducing kernel for *H*, denoted $K_w(z) = K(z, w)$, $\forall z, w \in \Omega$, if it satisfies the following two properties:

(i) $\forall w \in \Omega, K_w$ belongs to the Hilbert Space H, and
(ii) For all $w \in \Omega$ and $f \in H$, we have the reproducing property:

$$f(w) = \langle f, K_w \rangle_H$$

As a consequence, the reproducing kernel is Hermitian. Further details can be found in [2]. There is extensive research on the theory of reproducing kernels for spaces of analytic functions of the unit disk. However, the corresponding theory for the analytic spaces of the upper half plane is much less complete. In particular, in 2020, Bonyo [5] determined the reproducing kernels for the Hardy space of the upper half plane and the weighted Bergman space of the upper half plane, while Adhiambo [1], in 2020, partially determined the reproducing kernel for the classical Dirichlet spaces of the upper half plane reproducing kernel in addition to the growth condition for functions in $\mathcal{D}(\mathbb{U})$.

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Moreover, we extend the existing relationship between the reproducing kernels for the classical Bergman space and Dirichlet space of the unit disk to their equivalences on the upper half plane.

2. Preliminaries

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk. On $\mathbb{D}$, $dA(z)$ will represent the normalized area measure, and it is given as $dA(z) = \frac{1}{\pi} dxdy = \frac{r}{\pi} drd\theta$ for $z = x + iy = re^{i\theta}$. We can generalize the area measure defined on $\mathbb{D}$. In particular, for each $z \in \mathbb{D}$ and for $\alpha \in \mathbb{R}$, $\alpha > -1$, dm_α represents the weighted measure on $\mathbb{D}$, and it is given as $dm_\alpha(z) = (1-|z|^2)^\alpha dA(z)$. It is apparent that if $\alpha = 0$, $dm_0(z) = dA(z)$. Let $\mathbb{U} = \{w \in \mathbb{C} : Im(w) > 0\}$ denote the upper half plane, and $Im(w)$ is the imaginary part of *w*. On this set, $dA(w)$ will represent the area measure. Likewise, we can generalize the area measure defined on $\mathbb{U}$. In particular, for each $w \in \mathbb{U}$ and $\alpha > -1$, $d\mu_\alpha(w) = (Im(w))^\alpha dA(w)$ represents the weighted measure on $\mathbb{U}$. Also, for $\alpha = 0$, $d\mu_0(w) = dA(w)$. The Cayley transform $\psi(z) = \frac{i(1+z)}{1-z}$ maps the unit disk conformally to the upper half plane with the inverse $\psi^{-1}(w) = \frac{w-i}{w+i}$ mapping the upper half plane conformally onto the unit disk. For more details, we refer to [6, 8]. For an open set $\Omega \subseteq \mathbb{C}$, let $\mathcal{H}(\Omega)$ denotes the space of analytic or holomorphic functions $f : \Omega \to \mathbb{C}$, that is,

$$\mathcal{H}(\Omega) := \{f : \Omega \to \mathbb{C} : \text{ f is analytic}\}.$$

For more details, see [6, 8] and references therein.

Let $1 \leq p < \infty$, then we define the Hardy space of the unit disk, $H^p(\mathbb{D})$, as

$$H^p(\mathbb{D}) = \left\{ f \in \mathcal{H}(\mathbb{D}) : ||\ f||_{H^p(\mathbb{D})} = sup_{0<r<1} \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty \right\}$$

while the Hardy space of the upper half plane, $H^p(\mathbb{U})$, is given as

$$H^p(\mathbb{U}) = \left\{ f \in \mathcal{H}(\mathbb{U}) : ||\ f||_{H^p(\mathbb{U})} = sup_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{\frac{1}{p}} < \infty \right\}$$

As noted in [4], H^p-functions can be identified with their boundary values almost everywhere, and in that case, we have:

$$||\ f||_{H^p(\partial\mathbb{D})} = \left(\int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \quad (2.1)$$

and

$$||\ f||_{H^p(\partial\mathbb{U})} = \left(\int_{-\infty}^{\infty} |f(x)|^p dx \right)^{\frac{1}{p}} \quad (2.2)$$

For a comprehensive theory on Hardy spaces, we refer to [4, 5, 12].

For $1 \leq p < \infty$, $\alpha > -1$, we define the weighted Bergman space of the unit disk, $L_a^p(\mathbb{D}, m_\alpha)$, as

$$L_a^p(\mathbb{D}, m_\alpha) = \left\{ f \in \mathcal{H}(\mathbb{D}) : || f||_{L_a^p(\mathbb{D}, m_\alpha)} = \left(\int_{\mathbb{D}} |f(z)|^p dm_\alpha(z) \right)^{\frac{1}{p}} < \infty \right\}. \quad (2.3)$$

On the other hand, the weighted Bergman space of the upper half plane, $L_a^p(\mathbb{U}, \mu_\alpha)$, is given as

$$L_a^p(\mathbb{U}, \mu_\alpha) = \left\{ f \in \mathcal{H}(\mathbb{U}) : || f||_{L_a^p(\mathbb{U}, \mu_\alpha)} = \left(\int_{\mathbb{U}} |f(w)|^p d\mu_\alpha(w) \right)^{\frac{1}{p}} < \infty \right\}. \quad (2.4)$$

2.1.Remark

When $\alpha = 0$ in equation (2.3), we have the classical Bergman space of the unit disk. Consequently, when $\alpha = 0$ in equation (2.4), we get the classical Bergman space of the upper half plane.

Further details can be derived from [2, 5, 12].

For $\alpha > -1$, the weighted Dirichlet space of the unit disk, $\mathcal{D}_\alpha(\mathbb{D})$, is given as

$$\mathcal{D}_\alpha(\mathbb{D}) = \left\{ f \in \mathcal{H}(\mathbb{D}) : || f||_{\mathcal{D}_{\alpha,1}(\mathbb{D})} = \left(\int_{\mathbb{D}} |f'(z)|^2 dm_\alpha(z) \right)^{\frac{1}{2}} < \infty \right\}$$

where $|| \cdot ||_{\mathcal{D}_{\alpha,1}(\mathbb{D})}$ is a seminorm on $\mathcal{D}_\alpha(\mathbb{D})$.

There are two ways of transforming the seminorm, $|| \cdot ||_{\mathcal{D}_{\alpha,1}(\mathbb{D})}$, into a norm: In the first case, we have:

$$||f||^2_{\mathcal{D}_\alpha(\mathbb{D})} = |f(0)|^2 + ||f||^2_{\mathcal{D}_{\alpha,1}(\mathbb{D})} \quad (2.5)$$

and in the second case, we have:

$$|||f|||^2_{\mathcal{D}_\alpha(\mathbb{D})} = ||f||^2_{H^2(\partial\mathbb{D})} + ||f||^2_{\mathcal{D}_{\alpha,1}(\mathbb{D})}. \quad (2.6)$$

On the other hand, the weighted Dirichlet space of the upper half plane, $\mathcal{D}_\alpha(\mathbb{U})$, is given as:

$$\mathcal{D}_\alpha(\mathbb{U}) = \left\{ f \in \mathcal{H}(\mathbb{U}) : || f||_{\mathcal{D}_{\alpha,1}(\mathbb{U})} = \left(\int_{\mathbb{U}} |f'(w)|^2 d\mu_\alpha(w) \right)^{\frac{1}{2}} < \infty \right\}$$

where $||\cdot||_{\mathcal{D}_{\alpha,1}(\mathbb{U})}$ is a seminorm on $\mathcal{D}_\alpha(\mathbb{U})$.

Likewise, there are two ways of transforming the seminorm, $||\cdot||_{\mathcal{D}_{\alpha,1}(\mathbb{U})}$, into a norm.

In the first case, we have:

$$||f||^2_{\mathcal{D}_\alpha(\mathbb{U})} = |f(i)|^2 + ||f||^2_{\mathcal{D}_{\alpha,1}(\mathbb{U})} \quad (2.7)$$

and in the second case, we have:

$$|||f|||^2_{\mathcal{D}_\alpha(\mathbb{U})} = ||f||^2_{H^2(\partial\mathbb{U})} + ||f||^2_{\mathcal{D}_{\alpha,1}(\mathbb{U})}. \quad (2.8)$$

2.2. Remark

We see that there are two ways of defining the norm on the Dirichlet space. It is for this reason that we have two cases of the weighted Dirichlet space, which are $(\mathcal{D}_\alpha(.), ||.||_{\mathcal{D}_\alpha(.)})$ and $(\mathcal{D}_\alpha(.), |||.|||_{\mathcal{D}_\alpha(.)})$. In what follows, $(\mathcal{D}_\alpha(.), ||.||_{\mathcal{D}_\alpha(.)})$ will be denoted by $\mathcal{D}_\alpha(.)$ while $(\mathcal{D}_\alpha(.), |||.|||_{\mathcal{D}_\alpha(.)})$ by $\mathcal{D}_{*\alpha}(.)$.

When $\alpha = 0$, we have the classical Dirichlet spaces, which shall be denoted by $\mathcal{D}(.)$ and $\mathcal{D}_*(.)$. Since we have two cases of the classical Dirichlet space, that is, $\mathcal{D}(.)$ and $\mathcal{D}_*(.)$, we have different reproducing kernels for each of these spaces.

In particular, the reproducing kernel for $\mathcal{D}_*(\mathbb{D})$ is given, in [10], as

$$K^{\mathcal{D}_*}_{\mathbb{D}}(z,w) = \frac{1}{z\overline{w}} \log\left(\frac{1}{1-z\overline{w}}\right) \quad (2.9)$$

for $z, w \in \mathbb{D}$. On the other hand, the reproducing kernel for $\mathcal{D}_*(\mathbb{U})$, in [1], is defined as

$$K^{\mathcal{D}_*}_{\mathbb{U}}(z,w) = \frac{(z+i)(\overline{w}-i)}{(z-i)(\overline{w}+i)} \log \frac{i(z+i)(\overline{w}-i)}{2(z-\overline{w})}$$

for $z, w \in \mathbb{U}$. The reproducing kernel for $\mathcal{D}(\mathbb{D})$ is given in [3] as

$$K^{\mathcal{D}}_{\mathbb{D}}(z,w) = 1 + \log\left(\frac{1}{1-z\overline{w}}\right) \quad (2.10)$$

for $z, w \in \mathbb{D}$.

However, relative to $\mathcal{D}(.)$, the corresponding reproducing kernel on the upper half plane in addition to its properties are not known. More on Dirichlet spaces can be found in [1, 3, 7, 9, 10].

3. Reproducing kernel

for $\mathcal{D}(\mathbb{U})$.

Before determining the reproducing kernel for $\mathcal{D}(\mathbb{U})$, we first give some preliminary results. Let $\mathcal{D}(.)$ denote the classical Dirichlet Space and ψ the Cayley transform. The composition operator induced by ψ, C_ψ, is defined as $C_\psi f = f \circ \psi$, $\forall f \in \mathcal{D}(\mathbb{U})$ [1]. We determine some of the properties of the composition operator C_ψ in the following Proposition;

3.1 Proposition

Let $C_\psi : \mathcal{D}(\mathbb{U}) \to \mathcal{D}(\mathbb{D})$ be the composition by ψ operator. Then

(1) C_ψ *is a linear operator.*

(2) C_ψ *is bounded and hence continuous.*

(3) C_ψ *is an isometry from* $\mathcal{D}(\mathbb{U})$ *onto* $\mathcal{D}(\mathbb{D})$.

(4) C_ψ *is invertible with inverse* $C_\psi^{-1} = C_{\psi^{-1}}$.

(5) C_ψ *is an isomorphism.*

(6) C_ψ *is unitary.*

(7) C_ψ *is an analytic map of* $\mathcal{D}(\mathbb{U})$ *onto* $\mathcal{D}(\mathbb{D})$.

To prove (1), we see that $\forall \lambda_1, \lambda_2 \in \mathbb{C},\ f_1, f_2 \in \mathcal{D}(\mathbb{U}),\ z \in \mathbb{D}$,

$$\begin{aligned} C_\psi(\lambda_1 f_1 + \lambda_2 f_2)(z) &= ((\lambda_1 f_1 + \lambda_2 f_2) \circ \psi)(z) \\ &= (\lambda_1 f_1 + \lambda_2 f_2)(\psi(z)) \\ &= \lambda_1 f_1 \circ \psi(z) + \lambda_2 f_2 \circ \psi(z) \\ &= \lambda_1 C_\psi f_1(z) + \lambda_2 C_\psi f_2(z). \end{aligned}$$

as desired, which proves (1).

Next, we prove that C_ψ is bounded and hence continuous. We have that $f \in \mathcal{D}(\mathbb{U})$,

$$||C_\psi f|| = ||f \circ \psi|| \leq ||f||||\psi||$$

Taking sup on both sides over all $f \in \mathcal{D}(\mathbb{U})$ with $||f|| = 1$,

$$||C_\psi|| \leq ||\psi||$$

Thus, C_ψ is bounded, which implies that it is continuous. This proves (2).

For assertion (3), we prove that C_ψ is an isometry from $\mathcal{D}(\mathbb{U})$ *onto* $\mathcal{D}(\mathbb{D})$, that is,

$$||C_\psi f||_{\mathcal{D}(\mathbb{D})} = ||f||_{\mathcal{D}(\mathbb{U})}$$

Using the Cayley transform, we have that

$$w = \psi(\zeta) = \frac{i(1+\zeta)}{1-\zeta}$$

where $\zeta \in \mathbb{D}$ and $w \in \mathbb{U}$, and therefore,

$$i = \psi(0) = \frac{i(1+0)}{1-0}.$$

Also, we have

$$dA(w) = |\psi'(\zeta)|^2 dA(\zeta)$$

Therefore,

$$\begin{aligned} ||f||^2_{\mathcal{D}(\mathbb{U})} &= |f(i)|^2 + \int_{\mathbb{U}} |f'(w)|^2 dA(w) \\ &= |f(\psi(0))|^2 + \int_{\mathbb{D}} |f'(\psi(\zeta))|^2 |\psi'(\zeta)|^2 dA(\zeta) \\ &= |(f\circ\psi)(0)|^2 + \int_{\mathbb{D}} |(f\circ\psi)'(\zeta)|^2 dA(\zeta) \\ &= ||f\circ\psi||^2_{\mathcal{D}(\mathbb{D})}. \end{aligned}$$

Next, we prove that C_ψ is invertible with inverse $C_\psi^{-1} = C_{\psi^{-1}}$.

Let C_ψ be the composition by ψ operator and $C_{\psi^{-1}}$ be the composition by ψ^{-1} operator. So, for $f \in \mathcal{D}(\mathbb{U})$, $C_\psi f = f\circ\psi$, and for $g \in \mathcal{D}(\mathbb{D})$, $C_{\psi^{-1}} g = g\circ\psi^{-1}$.

As such, we have that, for $w \in \mathbb{U}$,

$$C_{\psi^{-1}}(C_\psi f)(w) = C_\psi f(\psi^{-1}(w)) = f(\psi(\psi^{-1}(w))) = f(w),$$

and, $\zeta \in \mathbb{D}$,

$$C_\psi(C_{\psi^{-1}} g)(\zeta) = C_{\psi^{-1}} g(\psi(\zeta)) = g(\psi^{-1}(\psi(\zeta))) = g(\zeta).$$

Therefore,

$$C_{\psi^{-1}}C_{\psi} = I = C_{\psi}C_{\psi^{-1}}$$
$$\Rightarrow C_{\psi}^{-1} = C_{\psi^{-1}},$$

as desired.

For (5), we wish to show that C_{ψ} is an isomorphism, that is, a surjective linear isometry. C_{ψ} is linear by (1). Also, C_{ψ} is isometric by (3). Moreover, C_{ψ} is invertible by (4), which implies that C_{ψ} is bijective thus surjective.

Next, we show that C_{ψ} is unitary, that is, $C_{\psi}^{*}C_{\psi} = C_{\psi}C_{\psi}^{*} = I$, whence

$C_{\psi}^{*} = C_{\psi}^{-1}$.

Taking note that $\mathcal{D}(.)$ is a Hilbert space, we have by the Riesz Representation Theorem that $(\mathcal{D}(.))^{*} \cong \mathcal{D}(.)$ and, therefore, $C_{\psi}^{*} : \mathcal{D}(\mathbb{D}) \to \mathcal{D}(\mathbb{U})$ since $C_{\psi} : \mathcal{D}(\mathbb{U}) \to \mathcal{D}(\mathbb{D})$.

Now, since C_{ψ} is an isometry by (3), we have that for all $f \in$ D(U),

$$\begin{aligned} ||C_{\psi}f||^{2}_{\mathcal{D}(\mathbb{D})} = ||f||^{2}_{\mathcal{D}(\mathbb{U})} \quad &\Leftrightarrow \quad \langle C_{\psi}f, C_{\psi}f\rangle_{\mathcal{D}(\mathbb{D})} = \langle f, f\rangle_{\mathcal{D}(\mathbb{U})} \\ &\Leftrightarrow \quad \langle C_{\psi}^{*}C_{\psi}f, f\rangle_{\mathcal{D}(\mathbb{U})} = \langle f, f\rangle_{\mathcal{D}(\mathbb{U})} \\ &\Leftrightarrow \quad C_{\psi}^{*}C_{\psi} = I. \end{aligned}$$

Thus

$$C_{\psi}^{*}C_{\psi} = I. \quad (3.1)$$

Using equation (3.1), for every $f \in$ D(U) and $g \in$ D(D), we obtain:

$$\begin{aligned} \langle C_{\psi}f, g\rangle_{\mathcal{D}(\mathbb{D})} &= \langle C_{\psi}f, C_{\psi}C_{\psi}^{-1}g\rangle_{\mathcal{D}(\mathbb{D})} \\ &= \langle C_{\psi}^{*}C_{\psi}f, C_{\psi}^{-1}g\rangle_{\mathcal{D}(\mathbb{U})} \\ &= \langle f, C_{\psi}^{-1}g\rangle_{\mathcal{D}(\mathbb{U})}, \end{aligned}$$

which implies that $C_{\psi}^{*} = C_{\psi}^{-1}$ and therefore $C_{\psi}C_{\psi}^{*} = I$, as desired.

Finally, we prove that C_{ψ} is an analytic map of $\mathcal{D}(\mathbb{U})$ *onto* $\mathcal{D}(\mathbb{D})$.

ψ is conformal, and so in particular it is analytic. Hence, the composition $f \circ \psi$ is a composition of two analytic functions, which, therefore, is again analytic, which completes the proof.

We can now determine the reproducing kernel for $\mathcal{D}(\mathbb{U})$.

3.2.Theorem

The reproducing kernel for $\mathcal{D}(\mathbb{U})$, *is*

$$K^{\mathcal{D}}_{\mathbb{U},\,w}(z) = 1 + \log\left(\frac{i(z+i)(\overline{w}-i)}{2(z-\overline{w})}\right) \quad (3.2)$$

$\forall z, w \in \mathbb{U}$.

Proof. Let $K^{\mathcal{D}}_{\mathbb{D}}$ and $K^{\mathcal{D}}_{\mathbb{U}}$ be the reproducing kernels for $\mathcal{D}(\mathbb{D})$ and $\mathcal{D}(\mathbb{U})$ respectively. The definition of the reproducing kernel for $\mathcal{D}(\mathbb{D})$ is given in equation (2.10). We need to work out the reproducing kernel for $\mathcal{D}(\mathbb{U})$.

The Cayley transform $\psi(z) = \frac{i(1+z)}{1-z}$ maps D onto U conformally with its inverse being $\psi^{-1}(w) = \frac{w-i}{w+i}$. C_ψ is the composition operator induced by ψ.

For each $\zeta \in \mathbb{D}$, and utilizing $K^{\mathcal{D}}_{\mathbb{D}}$,we have

$$\begin{aligned} C_\psi f(\zeta) = f(\psi(\zeta)) &= \langle C_\psi f, K^{\mathcal{D}}_{\mathbb{D},\,\zeta}\rangle_{\mathcal{D}(\mathbb{D})} \\ &= \langle f, C_\psi^{-1} K^{\mathcal{D}}_{\mathbb{D},\,\zeta}\rangle_{\mathcal{D}(\mathbb{U})} \end{aligned}$$

Now, we compute $C_\psi^{-1} K^{\mathcal{D}}_{\mathbb{D},\,\zeta}$. Letting $z \in \mathbb{U}$, we have

$$C_\psi^{-1} K^{\mathcal{D}}_{\mathbb{D},\,\zeta}(z) = K^{\mathcal{D}}_{\mathbb{D},\,\zeta}(\psi^{-1}(z))$$

Using the definition of $K^{\mathcal{D}}_{\mathbb{D}}$,

$$\begin{aligned} K^{\mathcal{D}}_{\mathbb{D},\,\zeta}(\psi^{-1}(z)) &= \left(\frac{1}{1-(\frac{z-i}{z+i})\overline{\zeta}}\right) \\ &= \left(\frac{1}{\frac{(z+i)-(z-i)\overline{\zeta}}{z+i}}\right) \\ &= \left(\frac{z+i}{(z+i)-(z-i)\overline{\zeta}}\right) \\ &= \left(\frac{z+i}{z+i-z\overline{\zeta}+i\overline{\zeta}}\right) \\ &= \left(\frac{z+i}{z(1-\overline{\zeta})+i(1+\overline{\zeta})}\right) \\ &= \left(\frac{z+i}{(1-\overline{\zeta})(z+i(\frac{1+\overline{\zeta}}{1-\overline{\zeta}}))}\right) \\ &= \left(\frac{z+i}{(1-\overline{\zeta})(z+\psi(\overline{\zeta}))}\right) \\ &= \left(\frac{z+i}{(1-\overline{\zeta})(z-\overline{\psi(\zeta)})}\right) \end{aligned}$$

We now have

$$C_\psi f(\zeta) = f(\psi(\zeta)) = \langle f, C_\psi^{-1} K^{\mathcal{D}}_{\mathbb{D},\, \zeta}\rangle_{\mathcal{D}(\mathbb{U})} = \left\langle f, 1 + \log\left(\frac{z+i}{(1-\overline{\zeta})(z-\overline{\psi(\zeta)})}\right)\right\rangle$$

As such,

$$f(\psi(\zeta)) = \left\langle f, 1 + \log\left(\frac{z+i}{(1-\overline{\zeta})(z-\overline{\psi(\zeta)})}\right)\right\rangle$$

And thus, taking $w = \psi(\zeta)$, for $w \in \mathbb{U}$, we have

$$f(w) = \left\langle f, 1 + \log\left(\frac{z+i}{(1-\frac{\overline{w}+i}{\overline{w}-i})(z-\overline{w})}\right)\right\rangle$$

Therefore,

$$\begin{aligned} K^{\mathcal{D}}_{\mathbb{U},\, w}(z) &= 1 + \log\left(\frac{(z+i)(\overline{w}-i)}{(\overline{w}-i-\overline{w}-i)(z-\overline{w})}\right) \\ &= 1 + \log\left(\frac{(z+i)(\overline{w}-i)}{-2i(z-\overline{w})}\right) \\ &= 1 + \log\left(\frac{i(z+i)(\overline{w}-i)}{2(z-\overline{w})}\right) \end{aligned}$$

As a consequence, we determine the norm of the reproducing kernel in $\mathcal{D}(\mathbb{U})$ given by equation (3.2).

3.3.Corollary

Let $K^{\mathcal{D}}_{\mathbb{U},\, w}$ be as given in equation (3.2). Then

$$||K^{\mathcal{D}}_{\mathbb{U},\, w}|| = \sqrt{\left(1 + \log\left(\frac{|w+i|^2}{4Im(w)}\right)\right)}. \tag{3.3}$$

Proof. We know that

$$||K^{\mathcal{D}}_{\mathbb{U},\, w}|| = \langle K^{\mathcal{D}}_{\mathbb{U},\, w}, K^{\mathcal{D}}_{\mathbb{U},\, w}\rangle^{\frac{1}{2}} = K^{\mathcal{D}}_{\mathbb{U},\, w}(w)^{\frac{1}{2}}$$

From equation (3.2), we can rewrite $K^{\mathcal{D}}_{\mathbb{U},\, w}(w)^{\frac{1}{2}}$ as:

$$\begin{aligned} K^{\mathcal{D}}_{\mathbb{U},\,w}(w)^{\frac{1}{2}} &= \left(1+\log\left(\frac{(w+i)(\overline{w}-i)}{-2i(w-\overline{w})}\right)\right)^{\frac{1}{2}} \\ &= \sqrt{\left(1+\log\left(\frac{|w+i|^2}{4Im(w)}\right)\right)}. \end{aligned}$$

Therefore,

$$||K^{\mathcal{D}}_{\mathbb{U},\,w}|| = \sqrt{\left(1+\log\left(\frac{|w+i|^2}{4Im(w)}\right)\right)},$$

as claimed. □

Now, we obtain the growth condition for functions in $\mathcal{D}(\mathbb{U})$.

3.4. Corollary

For every $f \in \mathcal{D}(\mathbb{U})$, we have

$$|f(w)| \leq ||f||\sqrt{\left(1+\log\left(\frac{|w+i|^2}{4Im(w)}\right)\right)} \tag{3.4}$$

where $||f|| = ||f||_{\mathcal{D}(\mathbb{U})}$.

Proof. Let $f \in \mathcal{D}(\mathbb{U})$. Then, by the reproducing property of $K^{\mathcal{D}}_{\mathbb{U},\,w}$, we have

that

$$f(w) = \langle f, K^{\mathcal{D}}_{\mathbb{U},\,w}\rangle$$

Now, by the Cauchy-Schwarz Inequality,

$$\begin{aligned} |f(w)| &= |\langle f, K^{\mathcal{D}}_{\mathbb{U},\,w}\rangle| \\ &\leq ||f||\,||K^{\mathcal{D}}_{\mathbb{U},\,w}||. \end{aligned}$$

As such,

$$|f(w)| \leq ||f||\,||K^{\mathcal{D}}_{\mathbb{U},\,w}||.$$

Therefore, using Corollary 3.3, we conclude that

$$|f(w)| \leq ||f|| \sqrt{\left(1 + \log\left(\frac{|w+i|^2}{4Im(w)}\right)\right)}$$

as desired. □

Finally, we give the relationship between the reproducing kernels for $\mathcal{D}(\mathbb{U})$ and $L^2_a(\mathbb{U})$.

3.5. Corollary

We have the relation $K_{\mathbb{U}}^{L^2_a}(z,w) = \partial_z \partial_{\overline{w}} K_{\mathbb{U}}^{\mathcal{D}}(z,w)$

Proof. We know that

$$K_{\mathbb{U}}^{L^2_a}(z,w) = \frac{1}{[-i(z-\overline{w})]^2} \quad (3.5)$$

while

$$K_{\mathbb{U}}^{\mathcal{D}}(z,w) = 1 + \log\left(\frac{i(z+i)(\overline{w}-i)}{2(z-\overline{w})}\right) \quad (3.6)$$

To confirm the relation between the reproducing kernels for the two spaces, we start by partially differentiating $K_{\mathbb{U}}^{\mathcal{D}}(z,w)$ with respect to $\overline{w}$.

$$\begin{aligned} \frac{\partial}{\partial \overline{w}}\left(1 + \log\left(\frac{i(z+i)(\overline{w}-i)}{2(z-\overline{w})}\right)\right) &= \frac{2(z-\overline{w})}{i(z+i)(\overline{w}-i)} \frac{\partial}{\partial \overline{w}}\left(\frac{i(z+i)(\overline{w}-i)}{2(z-\overline{w})}\right) \\ &= \frac{2(z-\overline{w})}{i(z+i)(\overline{w}-i)}\left[\frac{2i(z+i)((z-\overline{w})+(\overline{w}-i))}{(2(z-\overline{w}))(2(z-\overline{w}))}\right] \\ &= \left[\frac{(z-\overline{w})+(\overline{w}-i)}{(z-\overline{w})(\overline{w}-i)}\right] \end{aligned}$$

Next, we differentiate $\left[\frac{(z-\overline{w})+(\overline{w}-i)}{(z-\overline{w})(\overline{w}-i)}\right]$ partially with respect to z,

$$\begin{aligned} \frac{\partial}{\partial z}\left[\frac{(z-\overline{w})+(\overline{w}-i)}{(z-\overline{w})(\overline{w}-i)}\right] &= \left[\frac{(\overline{w}-i)((z-\overline{w})-(z-\overline{w})-(\overline{w}-i))}{(z-\overline{w})(\overline{w}-i)(z-\overline{w})(\overline{w}-i)}\right] \\ &= \frac{-1}{[z-\overline{w}]^2} \\ &= \frac{1}{[-i(z-\overline{w})]^2} \end{aligned}$$

which implies that

$$K_{\mathbb{U}}^{L^2_a} = \partial_z \partial_{\overline{w}} K_{\mathbb{U}}^{\mathcal{D}}(z,w).$$

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