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Modular Equivalents and Digital Roots of Irrational and Complex Numbers

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Abstract

Modular arithmetic, by definition, is arithmetic for integers. Since rational numbers are just ratios of two integers, the integral equivalents of rational numbers can be calculated using modular arithmetic for any modulus. But, the existing literature does not talk about integral equivalents of irrational numbers and complex numbers. This article proposes the idea of calculating them and finally proves a theorem related to an extension of Fermat's last theorem which could hold true only if the integral equivalents of irrational numbers and complex numbers existed. Also, it talks about a pattern that exists when the number of variables used in the extension of Fermat's last theorem is increased in a certain manner from 3 to infinity. Similarly, digital root (dr) is defined just for positive numbers and for non-terminating fractions. This article proposes the extension of its definition from positive integers to real numbers and even to complex numbers and thus calculates the digital root of irrational numbers and some complex numbers.

Keywords: Digital root; Integral equivalents; Irrational numbers; Complex numbers; Extension of Fermat's last theorem.

1. Introduction

Digital root of a positive number is a single-digit value obtained by continuous summation of its digits until a single-digit counting number is obtained [1, 2]. For example, the digits of 256 sums up to give 2+5+6=13, and the digits of 13 sums up to give 1+3=4. Here, 4 is the single digit. Therefore, the digital root of 256 is 4. Similarly, Modular Arithmetic, sometimes referred to as "clock arithmetic", is a system of arithmetic for integers done with a count that resets itself to zero every time a certain number is reached, called modulus [3, 4]. Since our calculation is based on base 10 and we have a total of 9 single-digit counting numbers, the digital root happens to be equivalent to arithmetic modulo 9 [1, 5].

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The only difference between them is the digital root of numbers that are exactly divisible by 9 is 9, but the numbers that are exactly divisible by 9 are congruent modulo 9 to 0 [1, 2].

2. Digital root of other rational numbers

Due to equivalence of digital root with modulo 9, we can extend the definition of digital root from positive integers to other rational numbers. For example, the digital root of -38 is 7 because 7 is the only single digit positive integer that is equivalent to -38 modulo 9 i.e. -38≡ -2≡ 7 (mod 9). The digital root of 0.47 is 2 because 2 is the only single digit whole number that is equivalent to 0.47 modulo 9 i.e. 0.47= 47/100 ≡ x (mod 9) or, 47 ≡ 100x (mod 9) or, 2 ≡ x (mod 9). Similarly, the digital root of 0.142857142857…. is 4. We can indeed calculate the digital roots of such non-terminating fractions, even though it seems almost impossible from the classical definition of digital root [5]. 0.1428571428 ... $= \mathbf{1/7} \equiv \boldsymbol{x}\ (\mathbf{mod\ 9})\ \ \boldsymbol{or}, \mathbf{1} \equiv \mathbf{7}\boldsymbol{x}\ (\mathbf{mod\ 9})$. Let's substitute 4 in the place of x and see if it satisfies. $\mathbf{1} \equiv \mathbf{7} \times \mathbf{4}\ (\mathbf{mod\ 9})\ \boldsymbol{or}, \mathbf{1} \equiv \mathbf{28} \equiv \mathbf{1}(\mathbf{mod\ 9})$, which is true. Digital roots of non-terminating fractions with multiple of 3 as their denominators don't exist because of their equivalence to arithmetic modulo 9 [5]. A number has to be co-prime with 9 to have multiplicative inverse modulo 9. For the reason that 3, 6 and 9 aren't co-primes with 9, their multiplicative inverses do not exist [6].

3. Digital root of Irrational numbers

Since modular arithmetic is defined just for integers, and rational numbers are always ratios of two integers [7], we could calculate the digital root of rational numbers easily. But to calculate the digital root of irrational numbers, we need to see if integral equivalents of irrational numbers modulo 9 exist. This paper proposes the idea of calculating integral equivalents of irrational numbers modulo 9 and thus calculates their digital root. Let $\sqrt{7} \equiv x$ (mod 9). After squaring, we get $7 \equiv x^2$ (mod 9) which is equivalent to $16 \equiv x^2$ (mod 9). Therefore, $x \equiv 4$ *or*, $-4 \rightarrow 5$ (mod 9). Similarly, $7 \equiv x^2$ (mod 9) is also equivalent to $25 \equiv x^2$ (mod 9). Therefore, $x \equiv 5$ or, $-5 \rightarrow 4$ (mod 9). Hence, the digital root of is 4 or, 5. There are two types of irrational numbers: algebraic and transcendental irrational numbers. We can calculate digital roots of algebraic ones quite easily as they are simply the roots of non-zero polynomials. Here are the steps one should follow: (i) Let the digital root of a given irrational number be x. (ii) Raise both sides to n^{th} power to cancel out n^{th} root. If it is square root, square on both sides as done in above example. (iii) Now, look for the perfect n^{th} powers of first nine digits that is/are congruent to the number (which is obtained after raising it to the n^{th} power modulo 9) and finally solve for the values of x: Just first nine perfect n^{th} powers because it takes maximum nine steps to repeat the pattern of digital root (See in Table 1). The following table shows how pattern of digital root repeats after every nine steps.

Table 1: Pattern of digital root repeating after every nine steps

N	N^2	dr(N^2)	N	N^2	dr(N^2)	N	N^2	dr(N^2)
1	1	1	7	49	4	13	169	7
2	4	4	8	64	1	14	196	7
3	9	9	9	81	9	15	225	9
4	16	7	10	100	1	16	256	4
5	25	7	11	121	4	17	289	1
6	36	9	12	144	9	18	324	9

The number of perfect n^{th} powers that are congruent to the number (which is obtained after raising it to the n^{th} power) modulo 9, within first nine perfect n^{th} powers, gives us the number of digital roots an irrational number can have. In the above example, 7 is congruent to 4^{th} and 5^{th} squares that are 16 and 25 respectively modulo 9. Therefore, $\sqrt{7}$ has two digital roots 4 or, 5. If any perfect powers that are congruent to the number (which is obtained by raising its power to the n^{th} degree) modulo 9 are not achieved, digital roots of such irrational numbers do not exist. Transcendental, along with some algebraic irrational numbers like π, $\sqrt{2}$, $\sqrt[3]{2}$ etc. do not have any digital roots. Using the above method, digital roots of some irrational numbers are calculated in the following examples.

a) Let, $\sqrt{5} \equiv x \pmod 9$. Then, $5 \equiv x^2 \pmod 9$. No perfect square (1, 4, 9, 16, 25, 36, 49, 64, 81, ...) is congruent to 5 modulo 9 at all. Therefore, $\sqrt{5}$ does not have any digital roots.

b) Let, $\sqrt{13} \equiv x \pmod 9$. Then, $13 \equiv x^2 \pmod 9$. Here, 13 is congruent to second and seventh perfect squares that are 4 and 49 respectively modulo 9. Therefore, the digital root of $\sqrt{13}$ is either 2 or, 7.

c) Let, $\sqrt[3]{10} \equiv x \pmod 9$. Then, $10 \equiv x^3 \pmod 9$. Here, 10 is congruent to 1, 64 and 343 which are first, fourth and 7^{th} perfect cubes respectively modulo 9. Therefore, the digital root of $\sqrt[3]{10}$ is either 1 or, 4 or, 7. Here, $x^3 \equiv 1 \pmod 9$ is treated like an equation $x^3 = 1$, but the real root is only taken in consideration because two other complex roots will have same integral equivalents as that of real roots produced from $x^3 \equiv 64 \pmod 9$ and $x^3 \equiv 343 \pmod 9$ (See in 3.2.).

d) Let, $\sqrt[5]{3203125} \equiv x$. Then, $3203125 \equiv x^5$. Here, 3203125 is congruent to the fourth perfect fifth that is 1024 modulo 9. Therefore, the digital root of $\sqrt[5]{3203125}$ is 4. Again, $x^5 \equiv 1024 \pmod 9$ produces five roots but other 4 complex roots do not have integral equivalents at all for modulus 9 (See in 3.2.). That is why we could get only one perfect fifth congruent to 3203125 modulo 9.

3.1. Integral equivalence of irrational numbers modulo k

Previously, we talked about just modulus 9 because the main motto of this article is to broaden the concept of digital root, but with a similar approach, the integral equivalents of irrational numbers modulo *k* (where, *k* is a natural number) can be found. Integral equivalents of some irrational numbers are calculated in the following examples:

a) $\sqrt{14} \pmod 5$: Let, $\sqrt{14} \equiv x \pmod 5$. Then, $14 \equiv x^2 \pmod 5$. Here, 14 is congruent to 4 and 9, which are second and third perfect squares respectively, modulo 5. Therefore, the integral equivalents of $\sqrt{14} \pmod 5$ is 2 or, 3. In fact, the integral equivalents of $\sqrt{14} \pmod 5$ is $5m+2$ or, $5m+3$ (where, *m* is an integer). The pattern of integral equivalents of perfect n^{th} powers for modulus 5 repeats maximum after every 5 steps. So, if the number (which is obtained after raising irrational number to the n^{th} power) is not congruent to any of the first five perfect n^{th} powers modulo 5, integral equivalents of such number does not exist for modulus 5.

b) $\sqrt[3]{15}$ (mod 7): Let, $\sqrt[3]{15} \equiv x$ (mod 7). Then, $15 \equiv x^3$ (mod 7). Here, 15 is congruent to 1, 8 and 64 modulo 7. Therefore, the integral equivalents of $\sqrt[3]{15}$ (mod 7) is 1 or, 2 or, 4 which implies the integral equivalents of $\sqrt[3]{15}$ (mod 7) is $7m$+1 or, $7m$+2 or, $7m$+4 (where, m is an integer). The pattern of integral equivalents of perfect nth powers for modulus 7 repeats maximum after every 7 steps. So if the number (which is obtained after raising irrational number to the nth power) is not congruent to any of the first seven perfect nth powers modulo 7, integral equivalents of such number does not exist for modulus 7. Here, $x^3 \equiv 1$ (mod 7) is treated like an equation $x^3 = 1$ but the real root is only taken in consideration because two other complex roots will have same integral equivalents as that of real roots produced from $x^3 \equiv 8$ (mod 7) and $x^3 \equiv 64$ (mod 7) (See in 3.2.)

c) π (mod 9): Let, $\pi \equiv x$ (mod 9). Raise to any power n, π^n is not going to be congruent to any perfect n^{th} powers. It will not be a whole number in the first place. That is why the integral equivalence of π (mod k) does not exist.

3.2. Integral equivalence of complex numbers modulo k

With a similar approach, integral equivalents of complex numbers can also be found. Let, $1+\sqrt{2}\,i \equiv x$ (mod 9). Then, $\sqrt{2}\,i \equiv x - 1$ (mod 9). After, squaring on both sides,

Or, $-2 \equiv (x-1)^2$ (mod 9)

Or, $7 \equiv (x-1)^2$ (mod 9)

Or, $\sqrt{7} \equiv (x-1)$ (mod 9)

Replacing $\sqrt{7}$ by its integral equivalents for modulus 9,

Or, 4 or, $5 \equiv (x-1)$ (mod 9)

$\therefore x \equiv 5$ or, 6 (mod 9)

Therefore, the digital root of $1+\sqrt{2}\,i$ is either 5 or, 6. In the above examples, we talked about complex roots and real roots. Let's see how that works. Let, $\sqrt[3]{15} \equiv x$ (mod 7). Then, $15 \equiv x^3$ (mod 7). Here, 15 is congruent to 1, 8 and 64 modulo 7. Therefore, the integral equivalents of $\sqrt[3]{15}$ (mod 7) is 1, 2 and 4. $x^3 \equiv 1$ (mod 7) is treated here like an equation $x^3 = 1$ but the real root is only taken in consideration because two other complex roots will have same integral equivalents as that of real roots produced from $x^3 \equiv 8$ (mod 7) and $x^3 \equiv 64$ (mod 7). The other two complex roots of $x^3 = 1$ are $\frac{-1 \pm \sqrt{-3}}{2}$. Now, let's find integral equivalents of these complex roots for modulus 7. Let

$\frac{-1+\sqrt{-3}}{2} \equiv x$ (mod 7)

Or, $-1+\sqrt{-3} \equiv 2x \pmod 7$

Or, $\sqrt{-3} \equiv 2x+1 \pmod 7$

Squaring on both sides,

$-3 \equiv (2x+1)^2 \pmod 7$

Or, $4 \equiv (2x+1)^2 \pmod 7$

Or, 2 or, $-2 \equiv 2x+1 \pmod 7$

Or, 2 or, $5 \equiv 2x+1 \pmod 7$

Or, 1or, $4 \equiv 2x \pmod 7$

Or, ½ or, $2 \equiv x \pmod 7$

The multiplicative inverse of 2 mod 7 is 4.

$\therefore x \equiv 2$ or, 4 (mod 7).

Similarly the integral equivalent of $\frac{-1-\sqrt{-3}}{2}$ is 4 or, 2.

On the other hand, $x^3 = 8$ and $x^3 = 64$ give 2 and 4 as their respective real roots. The other complex roots of $x^3 = 8$ will have 1 and 4 as their integral equivalents mod 7 and that of $x^3 = 64$ will have 1 and 2 as their integral equivalents mod 7. Now, this way we can know if integral equivalents of irrational numbers and complex numbers exist, and calculate them if they exist. Single-digit integral equivalents of numbers modulo 9 gives us their digital roots (except for multiple of 9) which we will be using to expand Fermat's last theorem from positive integers to broader set of numbers and state a complete new theorem.

4. Extension of Fermat's last theorem

Fermat's Last Theorem states that no three positive integers *a*, *b*, and *c* satisfy the equation $a^n + b^n = c^n$ for any integer value of *n* greater than 2 [8]. This theorem clearly cannot be extended to real numbers at all because take any positive real number *a* and *b*, *c* will be $\sqrt[n]{a^n + b^n}$, which is a real number for any integer value of *n*. However, if certain real numbers are excluded, the theorem may be extended to encompass a broader set of numbers. Surprisingly, if we exclude numbers with 3, 6 and 9 as their digital roots and numbers whose digital root do not exist, the theorem can be extended to real and even complex numbers. Let us divide every number that exists into two categories A and B. (i) Numbers of type A*:* Their digital roots lie in N_A = {1, 2, 4, 5, 7, 8}, (ii) Numbers of type B*:* Their digital roots lie in N_B = {3, 6, 9} or do not exist.

Statement: No three numbers *a*, *b* and *c* of type A satisfy the equation $a^n + b^n = c^n$ for any integer value of n other than {±1, ±5, ±7, ±11, ... 6*m*±1}. In other words, for n other than 6*m*±1, the equation $a^n + b^n = c^n$ needs at least one number from type B to hold true. But if we exclude numbers of type B, this equation has no number solution at all for $n \neq 6m\pm1$.

Below are the examples that clarify the statement:

a) $7^2 + 5^2 = (\sqrt{74})^2$: $\sqrt{74}$ is a type B number because it is equivalent to $\sqrt{2}$ (mod 9) and digital root of $\sqrt{2}$ does not exist.

b) $3^2 + 4^2 = 5^2$: 3 here lies in type B. This theorem also implies that every Pythagorean triple has a multiple of 3 in it. Otherwise, the Pythagoras theorem would not hold true if a number from type B were not used because *n*=2 does not lie in 6*m*±1.

c) $5^5 + 20^5 = (20.00394725.....)^5$: 20.00394725..... is a number from type A because 20.00394725..... = $\sqrt[5]{3203125} \equiv \sqrt[5]{7} \equiv \sqrt[5]{1024} \equiv \sqrt[5]{4^5} \equiv 4$ (mod 9). We could write this equation, even though every number used was from type A because n=5 lies in 6*m*±1.

d) $5^7 + 20^7 = (20.0001743816\ ...)^7$: 20.0001743816....= $\sqrt[7]{1280078125} \equiv \sqrt[7]{7} \equiv \sqrt[7]{823543} \equiv 7$ (mod 9). Since the integral equivalent of $20.0001743816\ ...$ does exist, the number is from type A. We could write this equation even though every number used was from type A because *n*=7 lies in 6*m*±1.

Now, here we state the proof of the extension of Fermat's last theorem that we have proposed. If $a^n + b^n = c^n$ stands true for numbers *a*, *b* and *c* of type A and just for *n* = 6*m*±1, $A'^n + B^n \equiv C^n$ (mod 9) should also stand true for *n* = 6*m*±1 where, A′, B and C are digital roots of *a*, *b* and *c* respectively which lie in N_A= {1, 2, 4, 5, 7, 8} as type A numbers are congruent to N_A modulo 9. Since, $A'^n + B^n \equiv C^n$(mod 9) shows same result maximum after every 6 integer values of *n* (see in table 2), the test can be run just for *n* = {1, 2, 3, 4, 5, 6}.

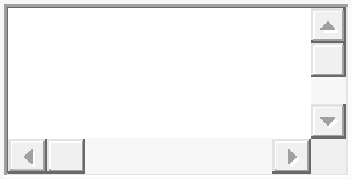
Table 1: Pattern of digital root repeating after every 6 consecutive integer values of n

X	dr(x)	x	dr(x)
2^1	2	2^7	2
2^2	4	2^8	4
2^3	8	2^9	8
2^4	7	2^{10}	7
2^5	5	2^{11}	5
2^6	1	2^{12}	1

If the extension of Fermat's last theorem holds, the result will be shown at 1 and 5 only.

[ ]

```
list_a = [1,2,4,5,7,8]

list_b = [1,2,3,4,5,6]

for a in list_a:

 for b in list_a:

   for c in list_a:

      for n in list_b:

        if (a**n+b**n)%9 == (c**n)%9:

          print("True at a={}, b={}, c={} and n={}".format(a,b,c,n))
```



True at a=1, b=1, c=2 and n=1

True at a=1, b=1, c=5 and n=5

True at a=1, b=4, c=5 and n=1

True at a=1, b=4, c=8 and n=5

True at a=1, b=7, c=2 and n=5

True at a=1, b=7, c=8 and n=1

True at a=2, b=2, c=1 and n=5

True at a=2, b=2, c=4 and n=1

True at a=2, b=5, c=4 and n=5

True at a=2, b=5, c=7 and n=1

True at a=2, b=8, c=1 and n=1

True at a=2, b=8, c=7 and n=5

True at a=4, b=1, c=5 and n=1

True at a=4, b=1, c=8 and n=5

True at a=4, b=4, c=2 and n=5

True at a=4, b=4, c=8 and n=1

True at a=4, b=7, c=2 and n=1

True at a=4, b=7, c=5 and n=5

True at a=5, b=2, c=4 and n=5

True at a=5, b=2, c=7 and n=1

True at a=5, b=5, c=1 and n=1

True at a=5, b=5, c=7 and n=5

True at a=5, b=8, c=1 and n=5

True at a=5, b=8, c=4 and n=1

True at a=7, b=1, c=2 and n=5

True at a=7, b=1, c=8 and n=1

True at a=7, b=4, c=2 and n=1

True at a=7, b=4, c=5 and n=5

True at a=7, b=7, c=5 and n=1

True at a=7, b=7, c=8 and n=5

True at a=8, b=2, c=1 and n=1

True at a=8, b=2, c=7 and n=5

True at a=8, b=5, c=1 and n=5

True at a=8, b=5, c=4 and n=1

True at a=8, b=8, c=4 and n=5

True at a=8, b=8, c=7 and n=1

With similar approach, the following can also be stated:

- No four numbers *a*, *b*, *c*, and *d* of type A satisfy the equation $a^n + b^n + c^n = d^n$ for any integer value of $n \neq 2m \pm 1$.
- No six numbers *a*, *b*, *c*, *d*, *e*, and *f* of type A satisfy the equation $a^n + b^n + c^n + d^n + e^n = f^n$ for any integer value of $n \neq 2m \pm 1$.
- No seven numbers *a*, *b*, *c*, *d*, *e*, *f*, and *g* of type A satisfy the equation $a^n + b^n + c^n + d^n + e^n + f^n = g^n$ for any integer value of $n \neq 6m \pm 1$.
- No nine numbers *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h* and *i* of type A satisfy the equation $a^n + b^n + c^n + d^n + e^n + f^n + g^n + h^n = i^n$ for any integer value of $n \neq 6m \pm 1$.

For increasing number of variables from 3 to infinity skipping $3m+2$ (where *m* is a set of positive integers), the equation shows a unique pattern (See in table 3).

Table 2: Repeating pattern of n for increasing number of variables from 3 to infinity in the equation

no. of variables	**Equations**	***n* is not equal to**
3	$a^n + b^n = c^n$	$6m \pm 1$
4	$a^n + b^n + c^n = d^n$	$2m \pm 1$
6	$a^n + b^n + c^n + d^n + e^n = f^n$	$2m \pm 1$
7	$a^n + b^n + c^n + d^n + e^n + f^n = g^n$	$6m \pm 1$
9	$a^n + b^n + c^n + d^n + e^n + f^n + g^n + h^n = i^n$	$6m \pm 1$
10	$a^n + b^n + c^n + d^n + e^n + f^n + g^n + h^n + i^n = j^n$	$2m \pm 1$
.	.	.
.	.	.
.	.	.
.	.	.

As they are based upon the fact that digital roots of irrational numbers and complex numbers can be calculated, extension of Fermat's last theorem and above results prove that the method discussed above to calculate integral equivalents of real and complex numbers for any modulus is correct.

5. Conclusion

We proposed the idea of calculating integral equivalents of irrational numbers and even complex numbers

for any modulus. Using the fact that arithmetic modulo 9 is equivalent to digital root to our advantage, we also calculated the digital roots of irrational numbers which were previously considered impossible because of their non-terminating and non-repeating decimal expansion. We also calculated the digital roots of complex numbers. Finally, we stated a theorem related to extension of Fermat's last theorem and a pattern for increasing numbers of variables which tells us that the integral equivalents of irrational numbers along with complex numbers do exist, and the methods to calculate them are correct.

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References

[1]. A. *Bonnie and* C. *Orin.* Problem Solving Through Recreational Mathematics. Dover Books on Mathematics (reprinted ed.). Mineola. NY: Courier Dover Publications, 1999, pp. 125–127.

[2]. F.M. Hall. An Introduction into Abstract Algebra, **1** (2nd ed.), Cambridge, U.K.: CUP Archive, 1980, p. 101

[3]. J.L. Berggren. "modular arithmetic". Encyclopædia Britannica.

[4]. M. Bullynck. "Modular Arithmetic before C.F. Gauss. Systematisations and discussions on remainder problems in 18th-century Germany"

[5]. T. Watkins. "Digit Sums for Repeating Decimals". applet-magic.com

[6]. E. W. Weisstein. "Modular Inverse". MathWorld.

[7]. K.H. Rosen. Discrete Mathematics and its Applications (6th ed.). New York, NY: McGraw-Hill. pp. 105, 158–160

[8]. W. W. R. Ball and Coxeter, H. S. M. Mathematical Recreations and Essays, 13th ed. New York: Dover, 1987, pp. 69-73.