Q-derivative of Modifie Tremblay Operator

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Abstract

A new class of analytic function defined by q-derivative of modified Tremblay fractional derivative operator in the unit disk was established. Coefficient bounds for \(|a_2|, |a_3|\) and \(|a_4|\) were obtained. Furthermore, Fekete - Szego estimate of functions belonging to the class \(S_{q,v,b,b}(z)\) was derived. The results obtained generalized some earlier ones in literature.

Keywords: Univalent function; Coefficient bounds; Tremblay operator.

1. Introduction

Let \(C, R\) and \(N\) denote the set of all complex numbers, real numbers and positive integers. The above shall be employed throughout the present investigation. Let \(\mathcal{F}(n)\) denote the class of functions \(f(z)\) which are normalized with the conditions

\[f(0) = f'(0) - 1 = 0\]

in the form

\[f(z) = z + c_{n+1}z^{n+1} + c_{n+2}z^{n+2} + \ldots, \quad n \in N,\]

and analytic in the open unit disk \(U = \{z \in C : |z| < 1\}\).

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Recently, some researchers developed interest in fractional derivative operator, particularly of order $u(0 \leq u < 1)$. This is likely due to its application in various field of endeavor.

1. Preliminaries

The well-known fractional derivative operator is given as

$$ D_v^k [z] = 0 \leq v < 1 $$

And it is presented integrally as

$$ D_v^k [z] = \frac{1}{\Gamma(1-v)} \int_0^z \frac{k(c)}{(z-c)^v} \, dc, 0 \leq v < 1 $$

(2.1)

Where $k(z)$ is an analytic function in a simply-connected complex domain containing the origin. The multiplicity of $(z-\varepsilon)^v$ can be removed by introducing $\log(z-\varepsilon)$ which is real whenever $z-\varepsilon > 0$. The $\Gamma$ is the popular gamma function. Let $k \in N_0$ and $0 \leq v < 1$; using (2.1) we obtain the Srivastava-Owa derivation of order $k + v$ given as

$$ D_v^{k+v} [z] = \frac{dz}{dv} D_v^k [z] $$

or

$$ D_v^{0+v} [z] = D_v^v [z] and D_v^{i+v} [z] = \frac{d}{dz} D_v^i [z] $$

Suppose $f(z) \in \chi(n)$ and of the form (1.1), the Tremblay operator denoted by $T_{i,v} [f(z)] = T_{i,v} [f(z)] or T_{i,s} [f]$ is defined as

$$ T_{i,s} [f(z)] = \frac{\Gamma(v)}{\Gamma(i)} z^{i-1} D^{i-v} z^{i-1} [f(z)] $$

(2.2)

For $0 < i < 1, 0 < v \leq 1, 0 \leq i - v < 1$ and $z \in U$, the operator $D_v^{i-v} z^{i-1} f(z)$ coincides with that of Srivastava-Owa operator earlier mentioned. From (2.2) the following results are obtainable:

$$ T_{i,s} [f(z)] = i \frac{z}{v} z + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+i)\Gamma(v)}{\Gamma(k+v)\Gamma(i)} a_k z^k $$

(2.3)
And

\[ D_q(T_{i,v}[f(z)]) = \frac{i}{v} z + \sum_{k=0}^{\infty} \frac{k}{\Gamma(k + i)\Gamma(v)} a_k z^{k-1} \] (2.4)

It is interesting to note that from all the available information due to (2.1)-(2.4) and taking cognizance of admissible values related to \( v \) and \( i \) then the following hold

\[ D_q(T_{i,v}[f(z)]) \in \mathcal{X}_n, \] (2.5)

\[ D_q(T_{i,v}[f(z)]) = f_0(z) \in \mathcal{X}_n, \] (2.6)

\[ D_q(T_{i,v}[f(z)]) = f(z) \in \mathcal{X}_n z \in U, \] (2.7)

In 2016, Esa et al. [4] modified the Tremblay operator and defined as:

\[
T^{v,i}_z f(z) = \frac{i}{v} T^{v,i}_z f(z) \\
T^{v,i}_z f(z) = \frac{\Gamma(i+1)}{\Gamma(v+1)} z^{1-i} D_z^{v-i} f(z)
\]

Simply we have

\[ T^{v,i}_z f(z) = z + \sum_{k=0}^{\infty} \frac{\Gamma(k+i)\Gamma(i+1)}{\Gamma(k+v)\Gamma(v+1)} a_k z^k \] (2.8)

This work was motivated by Irmark and Olga [10] and Esa et al. [4].

**Definition 2.1** Let \( \varphi(z) \in P \), \( 0 \leq i \leq 1, 0 \leq v \leq 1, 0 < q < 1, \beta \geq 0, s \geq 0, b \in C \setminus \{0\} \). A function \( T^{v,i}_z f(z) \in A \) of the form (2.8) is said to belong to the class \( S^{q,v,i,\beta,b}(z) \) if

\[ 1 + \frac{1}{b} \left( (1 + i \tan \beta) \frac{z D_q T^{v,i}_z f(z)}{T^{v,i}_z f(z)} - i \tan \beta - 1 \right) < \varphi(z) \]

**Remark 2.1** The following are some classes generalized by the class \( S^{q,v,i,\beta,b}(z) \). Let \( q \to 1 \),

1. for \( \varphi(z) = \frac{1 + A_z}{1 + B_z}, \beta = 0, b = 1 \) the class reduced to \( S^{\beta}(A, B) \) introduced and investigated by [16]
2. If \( \beta = 0, i = v = 1 \) the class reduce to starlike function

3. If \( \beta = 0, i = v = 1 \) the class reduce to class \( S^b(\varphi) \) introduced and studied by [12]

The useful lemma for this work is stated below:

**Lemma 2.1** If \( \omega \in \Omega \) then

\[
|\omega_2 - t\omega_1^2| \leq \max 1, |t|
\]

for any complex number \( t \). The result is sharp for the function \( \omega\zeta = z \) or \( \omega(z)z^2 \), (see also [9])

**Lemma 2.2** (Schwarz lemma)

Let the analytic function \( \varphi(z) \) be regular in \( U \) and let \( \varphi(0) = 0 \). If \( \varphi(z) \leq 1 \) in the open unit disk \( U \), then

\[
|\varphi(z)| \leq |z| \left( |z| < 1 \right) \quad \text{and} \quad |\varphi'(0)| \leq 1
\]

**Main Results**

**Theorem 3.1** Let \( \varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \ldots \) if \( T_{i,v}f(z) \) given by (2.8) belongs the class \( S^q_{v,i,\beta,b}(z) \) then,

\[
|a_2| \leq \frac{|b|B_1d_1\Gamma(2 + v)\Gamma(i + 1)}{(1 + i \tan \beta)([2]_q - 1)\Gamma(2 + i)\Gamma(v + 1)}
\]

\[
|a_3| \leq \frac{2|b|B_1d_1\Gamma(3 + v)\Gamma(i + 1)}{2(1 + i \tan \beta)([3]_q - 1)\Gamma(3 + i)\Gamma(v + 1)} \left( d_2 + \left( \frac{bb_1}{(1 + i \tan \beta)([2]_q - 1)} + \frac{bb_2}{4B_1} - \frac{1}{2} \right) d_1^2 \right)
\]

**Proof:**

Suppose \( T_{i,v}f(z) \in S^q_{v,i,\beta,b}(z) \) by definition

\[
1 + \frac{1}{b} \left( 1 + i \tan \beta \right) \frac{z^2 T_{i,v}f(z)}{T_{i,v}f(z)} - i \tan \beta - 1 < \varphi(z)
\]

Therefore,
\[ 1 + \frac{1}{b} \left( (1 + i \tan \beta) \frac{z D_q T_{i,v} f(z)}{T_{i,v} f(z)} - i \tan \beta - 1 \right) = \varphi(\omega(z)) \]  

(3.1)

Now,

\[ \frac{z D_q T_{i,v} f(z)}{T_{i,v} f(z)} = z + \sum_{k=2}^{\infty} \frac{k}{q} \frac{\Gamma(k+i)\Gamma(v+1)}{\Gamma(v+i)\Gamma(i+1)} a_k z^k = \frac{1}{1 + \sum_{k=2}^{\infty} \frac{k}{q} \frac{\Gamma(k+i)\Gamma(v+1)}{\Gamma(v+i)\Gamma(i+1)} a_k z^{k-1}} \]

\[ 1 + \frac{1}{b} \left( (1 + i \tan \beta) \frac{z D_q T_{i,v} f(z)}{T_{i,v} f(z)} - i \tan \beta - 1 \right) = 1 + \frac{(1 + i \tan \beta) \left( [2]_q - 1 \right) \frac{\Gamma(2+i)^2 \Gamma(v+1)^2}{\Gamma(2+v)\Gamma(i+1)^2} a_2 z \]

\[ + \frac{(1 + i \tan \beta)}{b} \left( \left( [3]_q - 1 \right) \frac{\Gamma(3+i)^2 \Gamma(v+1)^2}{\Gamma(3+v)\Gamma(i+1)^2} a_3 - \left( [2]_q - 1 \right) \frac{\Gamma(2+i)^2 \Gamma(v+1)}{\Gamma(2+v)\Gamma(i+1)^2} a_2^2 \right) z^2 \]

\[ + \frac{(1 + i \tan \beta)}{b} \left( \left( [4]_q - 1 \right) \frac{\Gamma(4+i)^2 \Gamma(v+1)^2}{\Gamma(4+v)\Gamma(i+1)^2} a_4 - \left( [3]_q - [2]_q \right) \frac{\Gamma(2+i)^2 \Gamma(3+i)^2 \Gamma(v+1)^2}{\Gamma(2+v)\Gamma(3+v)\Gamma(i+1)^2} a_2 a_3 \right) z^3 + \ldots \]

Let,

\[ P(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = \frac{1}{2} \left( d_1 z + \left( d_2 - \frac{d_1^2}{2} \right) z^2 + \ldots \right) \]

(3.2)

Now,

\[ \varphi(\omega(z)) = 1 + \frac{B_1 d_1}{2} z + \left( \frac{B_1 d_2}{2} + \frac{(B_2 - B_1) d_1^2}{4} \right) z^2 + \frac{B_1 d_3}{2} \left( \frac{B_2 - B_1}{2} \right) d_2 + \frac{B_1 - 2B_2 + B_3 d_1^3}{8} \right) z^3 + \ldots \]

Since,

\[ 1 + \frac{1}{b} \left( (1 + i \tan \beta) \frac{z D_q T_{i,v} f(z)}{T_{i,v} f(z)} - i \tan \beta - 1 \right) = \varphi(\omega(z)) \]

We have

\[ 1 + \frac{(1 + i \tan \beta)}{b} \left( [2]_q - 1 \right) \frac{\Gamma(2+i)^2 \Gamma(v+1)^2}{\Gamma(2+v)\Gamma(i+1)^2} a_2 z \]

\[ + \frac{(1 + i \tan \beta)}{b} \left( \left( [3]_q - 1 \right) \frac{\Gamma(3+i)^2 \Gamma(v+1)^2}{\Gamma(3+v)\Gamma(i+1)^2} a_3 - \left( [2]_q - 1 \right) \frac{\Gamma(2+i)^2 \Gamma(v+1)}{\Gamma(2+v)\Gamma(i+1)^2} a_2^2 \right) z^2 \]
\begin{align*}
&+ \left( \frac{1 + i \tan \beta}{b} \right) \left( \left[ 4 \right]_q - 1 \right) \frac{\Gamma(4 + i) \Gamma(v + 1)}{\Gamma(4 + v) \Gamma(i + 1)} a_4 - \left[ 2 \right]_q \frac{\Gamma(2 + i) \Gamma(3 + i) \Gamma(v + 1)^2}{\Gamma(2 + v) \Gamma(3 + v) \Gamma(i + 1)} a_2 a_3 \right) z^3 + \ldots \\
&= 1 + \frac{B_1 d_1}{2} z + \left( B_1 d_2 + \frac{(B_2 - B_1) d_1^2}{4} \right) z^2 + \left( B_4 d_3 + \frac{(B_2 - B_1) d_2 d_1}{2} + \frac{B_1 - 2B_2 + B_4 d_1^3}{8} \right) z^3 + \ldots
\end{align*}

Comparing coefficients of \( z, z^2, z^3 \), we have

\begin{align*}
|a_z| &\leq \frac{|b| |B_1 d_1| \Gamma(2 + v) \Gamma(i + 1)}{\left( 1 + i \tan \beta \right) \left[ 2 \right]_q - 1} \Gamma(2 + i) \Gamma(v + 1) \\
|a_3| &\leq \frac{2 |b| |B_1 d_1| \Gamma(3 + v) \Gamma(i + 1)}{2\left( 1 + i \tan \beta \right) \left[ 3 \right]_q - 1} \Gamma(3 + i) \Gamma(v + 1) \left( d_2 + \left( \frac{bB_1}{1 + i \tan \beta} \left[ 2 \right]_q - 1 \right) + \frac{bB_2 - 1}{2} \right) \right) d_1^2)
\end{align*}

Which completes the proof.

**Corollary 3.1** Suppose \( T_{i,v} f(z) \) belongs to the class \( S^{q,1,0,\beta,b}(z) \) we obtained

\begin{align*}
|a_z| &\leq \frac{|b| |B_1 d_1|}{\left( 1 + i \tan \beta \right) } \text{ and} \\
|a_3| &\leq \frac{2B_1}{2\left( 1 + i \tan \beta \right) \left[ 3 \right]_q - 1} \Gamma(3 + i) \Gamma(v + 1) \left( \frac{bB_1}{1 + i \tan \beta} + \frac{bB_2 - 1}{2} \right) d_1^2
\end{align*}

**Remark 3.1** This is the result obtained in [12]

**Theorem 3.2** Let \( T_{i,v} f(z) \) be the function given by (2.8) and belongs to the class \( S^{q,1,\beta,b}(z) \) then for any complex number \( v \in C \)

\begin{align*}
|a_3 - va_z|^2 &\leq \frac{2 |b| |B_1| \Gamma(3 + v) \Gamma(i + 1)}{2\left( 1 + i \tan \beta \right) \left[ 3 \right]_q - 1} \Gamma(3 + i) \Gamma(v + 1) \max \{1, v\}
\end{align*}

where,

\begin{align*}
v = \left( \frac{bB_1}{1 + i \tan \beta} + \frac{bB_2}{4B_1} - \frac{1}{2} \left( \frac{2B_1 \Gamma(3 + i) \Gamma(v + 1) \Gamma(2 + v)^2 \Gamma(i + 1)^2}{\left( 1 + i \tan \beta \right) \left[ 3 \right]_q - 1} \right) \right)
\end{align*}

**Proof:**
Let $T_{v, f}(z) \in S^{v, T, \beta, b}(z)$ then substituting (3.3) and (3.4) into $|a_3 - ka_2^2|$ we have

$$a_3 - va_2^2 \leq \frac{2bB_1 \Gamma(3 + v) \Gamma(i + 1)}{2(1 + i \tan \beta) \left[3^aq - 1\right] \Gamma(3 + i) \Gamma(v + 1)} \left(d_2 \left(\frac{bB_1}{2(1 + i \tan \beta) \left[2^aq - 1\right]} + \frac{bB_2}{4B_1} - \frac{1}{2}\right) d_1 \right)$$

$$- v \frac{b^2 B_1^2 d_2 \Gamma(2 + v)^2 \Gamma(i + 1)^2}{(1 + i \tan \beta)^2 \left[2^aq - 1\right]^2 \Gamma(2 + i)^2 \Gamma(v + 1)^2}$$

then,

$$|a_3 - va_2^2| = \frac{2bB_1 \Gamma(3 + v) \Gamma(i + 1)}{2(1 + i \tan \beta) \left[3^aq - 1\right] \Gamma(3 + i) \Gamma(v + 1)} \left(d_2 - vd_1 \right)$$

$$|a_3 - va_2^2| \leq \frac{2bB_1 \Gamma(3 + v) \Gamma(i + 1)}{2(1 + i \tan \beta) \left[3^aq - 1\right] \Gamma(3 + i) \Gamma(v + 1)} \left(d_2 - vd_1 \right)$$

Using lemma (2.1) in equation (3.6), we have

$$|a_3 - va_2^2| \leq \frac{2bB_1 \Gamma(3 + v) \Gamma(i + 1)}{2(1 + i \tan \beta) \left[3^aq - 1\right] \Gamma(3 + i) \Gamma(v + 1)} \max \left\{1, \sqrt{v}\right\}$$

where

$$v = \sqrt{\left(\frac{bB_1}{2(1 + i \tan \beta) \left[2^aq - 1\right]} + \frac{bB_2}{4B_1} - \frac{1}{2}\right) \Gamma(2 + i)^2 \Gamma(i + 1)^2 + \frac{2bB_1 \Gamma(3 + v) \Gamma(i + 1)^2}{(1 + i \tan \beta) \left[3^aq - 1\right] \Gamma(3 + i) \Gamma(v + 1)^2}}$$

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