International Journal of Formal Sciences: Current and Future Research Trends (IJFSCFRT)
ISSN (Print) , ISSN (Online)
© International Scientific Research and Researchers *Association*
https://ijfscfrtjournal.isrra.org/index.php/Formal_Sciences_Journal/index

On Divisibility and $\gamma -$ Periodicity in the Quasigeometric Fibonacci Sequence

Lovemore Mamombe*

Independent Researcher, Harare

Email: mamombel@gmail.com

Abstract

The quasigeometric Fibonacci sequence $F_n \begin{cases} f_1 = 1, f_2 = 2 \\ f_{n+2} = f_{n+1} + f_n, n \geq 1 \end{cases}$ has interesting divisibility properties, namely that $\frac{f_{2nj+2j-n-2}}{f_n}, j \geq 1, n \geq 2$ and $\frac{f_{2nj+2j-1}}{f_n}, j \geq 1, n \geq 2$ are integers. Results of this research show that $\frac{f_{2nj+2j-n-2}}{f_n} = h_1, j \geq 1, n \geq 2$ and $\frac{f_{2nj+2j-1}}{f_n} = g_{n-2}, j \geq 1, n \geq 3$ where H_n and G_n are golden section quasigeometric sequences. This phenomenon of periodicity is herein named $\gamma -$periodicity. Concept is important in profiling the behavior of F_n and the golden section and may be invoked in science and technology development.

Keywords: $\gamma -$periodicity; Fibonacci sequence; golden section quasigeometric sequence; quasigeometric Fibonacci sequence.

1. Introduction

The sequence

$$F_n \begin{cases} f_1 = 1, f_2 = 1 \\ f_{n+2} = f_{n+1} + f_n, n \geq 1 \end{cases} \tag{1.1}$$

and the irrational number

$$\varphi = \frac{1+\sqrt{5}}{2} \tag{1.2}$$

areknown as the Fibonacci sequence and the golden section respectively.

----

* Corresponding author.

These two concepts are inseparable for

$$\lim_{n\to\infty}\frac{f_{n+1}}{f_n}=\varphi \tag{1.3}$$

The cropping up of Fibonacci numbers and therefore the golden section in nature especially in the botanical phenomenon of phyllotaxis and in human DNA [1-10] justifies the investment directed towards research of these concepts. Nature has already applied the concepts and it remains for the human race or some other sufficiently intelligent and interested race to investigate these concepts in order to understand how the universe works and to apply the concepts in science and technological development.

Consider

$$H_n\begin{cases} h_{n+1}=round(\varphi h_n), n\geq 1 \\ h_2-h_1=h_0 \\ round(\varphi h_0)\neq h_1 \\ h_{n+2}=h_{n+1}+h_n, n\geq 1\end{cases} \tag{1.4}$$

H_n is called the golden section quasigeometric sequence [10,11]. The quasigeometric Fibonacci sequence

$$F_n=1,2,3,5,8,\ldots \tag{1.5}$$

is defined by Relation (1.4). The object of this manuscript is to define and study γ −periodicity in the sequence (1.5). Hereinafter the designation F_n is reserved for the sequence (1.5). H_n, G_nor any sequence designated by some other letter is defined by the Relation (1.4).

2. Results

The sequence $\mathrm{F_n}$ has interesting divisibility properties, namely that

$$\frac{\mathrm{f_{2nj+2j-n-2}}}{\mathrm{f_n}}, \mathrm{j}\geq 1, \mathrm{n}\geq 2$$

and

$$\frac{\mathrm{f_{2nj+2j-1}}}{\mathrm{f_n}}, \mathrm{j}\geq 1, \mathrm{n}\geq 2$$

are integers. We analyze these integers and show by way of theorems 2.1 and 2.2 that they have important properties when viewed as elements of golden section quasigeometric sequences. This phenomenon of periodicity is herein named γ −periodicity.

Theorem 2.1

$$\frac{f_{2nj+2j-n-2}}{f_n} = h_1, j \geq 1, n \geq 2 \tag{2.1}$$

Proof

For $2 \leq n \leq 3$ we have that

$$\frac{f_{2nj+2j-n-2}}{f_n} = \sum_{r=1}^{2j-1} (-1)^{r+1} f_{(n+1)r-n}$$

For $n \geq 4$ we have two scenarios.

Scenario I: even $n \geq 4$

For odd $j \geq 1$ we have that

$$\frac{f_{2nj+2j-n-2}}{f_n} = f_{(2n+2)j-2n-1} + \sum_{r=1}^{\frac{j-1}{2}} \sum_{u=1}^{2} f_{(4n+4)r+(2n-2)u-6n-1} - \sum_{r=1}^{\frac{j-1}{2}} \sum_{u=1}^{2} f_{(4n+4)r+(2n-2)u-4n+1}$$

For even $j \geq 2$ we have that

$$\frac{f_{2nj+2j-n-2}}{f_n} = f_{(2n+2)j-2n-1} + \sum_{r=1}^{\frac{j-2}{2}} \sum_{u=1}^{2} f_{(4n+4)r+(2n-2)u-4n+1} - \sum_{r=1}^{\frac{j}{2}} \sum_{u=1}^{2} f_{(4n+4)r+(2n-2)u-6n-1}$$

Scenario II: odd $n \geq 5$

For $j \geq 1$ we have that

$$\frac{f_{2nj+2j-n-2}}{f_n} = \sum_{r=1}^{j-1} \sum_{u=1}^{2} (-1)^u f_{(2n+2)r+4u-7} + 1$$

Theorem 2.7 in [10] completes the proof.

Theorem 2.2

$$\frac{f_{2nj+2j-1}}{f_n} = h_{n-2}, j \geq 1, n \geq 3 \tag{2.2}$$

Proof

For $n = 3$ we have that

$$\frac{f_{8j-5}}{f_3}=\sum_{r=1}^{2j}(-1)^r f_{4r-3}$$

For $n \geq 4$ we have two scenarios.

<u>Scenario I: even $n \geq 4$</u>

For odd $j \geq 1$ we have that

$$\frac{f_{2nj+2j-1}}{f_n}=f_{(2n+2)j-n}+\sum_{r=1}^{\frac{j-1}{2}}\sum_{u=1}^{2}f_{(4n+4)r+(2n-2)u-5n}-\sum_{r=1}^{\frac{j-1}{2}}\sum_{u=1}^{2}f_{(4n+4)r+(2n-2)u-3n+2}-f_{n-2}$$

For even $j \geq 2$ we have that

$$\frac{f_{2nj+2j-1}}{f_n}=f_{(2n+2)j-n}+\sum_{r=1}^{\frac{j-2}{2}}\sum_{u=1}^{2}f_{(4n+4)r+(2n-2)u-3n+2}-\sum_{r=1}^{\frac{j}{2}}\sum_{u=1}^{2}f_{(4n+4)r+(2n-2)u-5n}+f_{n-2}$$

<u>Scenario II: odd $n \geq 5$</u>

For $j \geq 1$ we have that

$$\frac{f_{2nj+2j-1}}{f_n}=\sum_{r=1}^{j}\sum_{u=1}^{2}(-1)^u f_{(2n+2)r+4u-n-8}$$

Theorem 2.7 in [10] completes the proof.

Remark 2.1

Notice that when $n = 2$ Theorem 2.2 is stated as

$$\frac{f_{2nj+2j-1}}{f_n}=h_{n-2}=h_0 \qquad (2.3)$$

This statement is true but the straightforward statement

$$\frac{f_{2nj+2j-1}}{f_n}=h_1 \qquad (2.4)$$

is consistent with Theorem 2.1. Lemma 2.1 shows how Equation (2.3) is correct.

Lemma 2.1

$$g_o = h_n, 1 \leq n \leq 2 \tag{2.5}$$

Proof

By definition

$g_0 = g_2 - g_1$

Let

$g_0 = h_n$

But

$g_1 = q_m \pm 1, g_2 = q_{m+1} \pm 2$

so that

$h_n = q_{m-1} \pm 1$

<u>Scenario I:</u>$g_1 = q_4 + 1$

From Theorem 2.5 in [10] this means that

$q_m = p_{m+r-1} - f_m, r \geq 4$

$\therefore h_n = p_{r+2} - 2 \text{ or } p_{r+2} - 4$

And it follows that $n = 1$.

<u>Scenario II:</u>$g_1 = q_4 - 1$

From Theorem 2.6 in [10] this means that

$q_m = p_{m+r-1} + f_m, r \geq 4$

$\therefore h_n = p_{r+2} + 2 \text{ or } p_{r+2} + 4$

And it follows that $n = 1$.

<u>Scenario III:</u> $m = 5$

$h_n = q_4 \pm 1$

It follows that $\mathrm{n} = 1$ or 2.

Scenario IV: $\mathrm{m} \geq 6$

$\mathrm{h_n} = \mathrm{q_{m-1}} \pm 1$

It follows from theorems 2.7 in [10] that $\mathrm{n} = 1$.

Proof is completed.

3. Conclusion

Various forms of periodicity in the sequence (1.1) which also apply to F_n have been studied by other researchers. This manuscript therefore shows one more important kind of periodicity exhibited by F_n. Results show that the divisibility properties of F_n link this sequence with other golden section quasigeometric sequences systematically. Concept is important in revealing some hidden behaviors of F_n and the golden section and may find applications in many areas of science and technology.

References

[1]. Elliot MA, Kelly J, Friedel J, Brodsky J, Mulcahy P. The golden section as optical limitation.PloS ONE. 2015;10(7):1-15.

[2]. Navon D. The sisters of the golden section. Perception. 2011;40:705-24.

[3]. Thapa GB, Thapa R. The relation of golden ratio, mathematics and aesthetics. Journal of the Institute of Engineering. 2018;14(1):188-99.

[4]. Phillips ME. Rethinking the role of the golden section in music and music scholarship. Creativity Research Journal. 2019;31(4):419-27.

[5]. Wares A. The golden ratio and regular hexagons. International Journal of Mathematical Education in Science and Technology. 2020;51(1):157-60.

[6]. Widrick RM, Raskin JD. Age-related stigma and the golden section hypothesis. Aging and Mental Health. 2010;14(4):375-85.

[7]. Raskin JD, Brett BL. Does the reverse golden section hold? Journal of Constructivist Psychology. 2014;27(2):137-46.

[8]. Nikolic ST, CosicI, Pecujlija M, Miletic A. The effect of the ‘golden ratio’ on consumer behaviour. African Journal of Business Management. 2011;5(20):8347-60.

[9]. Saraf S, Saraf P. The golden proportion: key to the secret of beauty. The Internet Journal of Plastic Surgery. 2013;9(1):1-16.

[10]. Mamombe L. Proportiones perfectus law and the physics of the golden section. Asian Research Journal of Mathematics. 2017;7(1):1-41.

[11]. Mamombe L. Golden section and evolving systems. Journal of Scientific Research and Reports. 2019;22(6):1-8.