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Regularization Methods for an Ill-Posed Elliptic Problem

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Abstract

We study an abstract elliptic Cauchy problem associated with an unbounded self-adjoint positive operator, which has a continuous spectrum. It is well-known that such a problem is severely ill-posed; that is, the solution does not depend continuously on the Cauchy data.

Keywords: Regularization methods; Ill-Posed Elliptic Problem.

1. Introduction

We propose two spectral regularization methods to construct an approximate stable solution to our original problem.

Finally, some other convergence results including some explicit convergence rates are also established under a priori bound assumptions on the exact solution. But other methods can be used in our case here. The complexity of studying a poorly posed problem requires mastery of certain concepts, especially in elliptical case.

2. Regularization And Error Estimates

2.1. The Truncation Method

From

$$u(y) = U(y) + W(y) = \frac{1}{2}\int_{\gamma}^{+\infty}\left(e^{y\sqrt{\lambda}} + e^{-y\sqrt{\lambda}}\right) dE_{\lambda}f. \qquad (1'')$$

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We can see that the term $e^{y\sqrt{\lambda}}$ *is the cause of unstability. In order to overcome the ill-posedness of problem:*

$$u_{yy}(y) = Au(y), \qquad 0 < y < L,$$

$$u(0) = f, \qquad (1')$$

$$u_y(0) = 0,$$

we modify the solution by filtering the high frequencies using a suitable method and instead consider (1'') only for $\lambda \le \beta(\delta)$, *where* $\beta(\delta)$ *is some constant which satisfies* $\lim_{\delta} \to \beta(\delta) = +\infty$.

According to spectral theory of self-adjoint operators [20],*for any bounded Borel set,* $\Delta_\beta = \{\gamma \le t \le \beta\} \subseteq \sigma(A) = [\gamma, +\infty[$, *we can define the orthogonal projection*

$$\mathbf{1}_{\Delta_\beta} = \int_\gamma^{+\infty} \mathbf{1}_{\Delta_\beta}(\lambda) dE_\lambda = E_{\beta,} \qquad (1)$$

$$\forall h \in H, \qquad h_\beta = E_\beta h \to h, \;\; \beta \to +\infty.$$

To solve (1) *in a stable way we approximate* f *by its projection* $f_{\beta,}$ *and instead of considering* (1) *with* f *we take its projected version*

$$u_\beta(x) = cosh\left(y\sqrt{A}\right) f_\beta \qquad \mathbf{(2)}$$

$$=\frac{1}{2}\int_\gamma^{+\infty} \left(e^{y\sqrt{\lambda}} + e^{-y\sqrt{\lambda}}\right) \mathbf{1}_{[\gamma,\beta]} dE_\lambda f,$$

Where $\mathbf{1}_{[a,b]}$ *is the characteristic function of the interval* $[a, b]$ *for* $a < b$. *The quantity* β *is referred to as a cut-off frequency. Let* f $(resp., f_\delta)$ *be the exact* $(resp., the\ measured\ data)$ *at y=0, such that* $\|f - f_\delta\| \le \delta$.

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The approximated solution v_β^δ *corresponding to the measured data* f_δ *is denoted by*

$$v_\beta^\delta(y) = \frac{1}{2}\int_y^{+\infty} \left(e^{y\sqrt{\lambda}} + e^{-y\sqrt{x}}\right) \mathbf{1}_{[\gamma,\beta,]} dE_\lambda f_{\delta.} \qquad (3)$$

For simplicity, we denote the solution of problem (1') by $u(y)$, *and the regularized solution associated to the data* f_δ *by* $v_\beta^\delta(y)$.

Our first main theorem is the following theorem.

Theorem 1: *The solution defined in* (2) *depends continuously in* $\boldsymbol{C}\ ([\boldsymbol{O},\boldsymbol{L}],\boldsymbol{H})$ *on the data* f; *that is, if* u_β^1 *and* u_β^2 *are two regularized solutions corresponding to* f_1 *and* f_2, *respectively, then on has*

$$\left\|u_\beta^1(y)-u_\beta^2(y)\right\| \le e^{y\sqrt{\beta}}\|f_1-f_2\|. \qquad (4)$$

This inequality implies that the solution of the regularized problem (2) depends continuously on the data f*.*

Now we compute the difference between the original solution $u=u(y;f)$ *and the approximate solution* $v_\beta^\delta = v_\beta^\delta(y;f_\delta)$.

Theorem 2: *Let* $u\in C([O;L];H)$ *be a solution problem* $(1')$ *with the exact data* $f\in H$; *then the following estimate holds:*

$$\left\|u(y)-u_\beta(y)\right\| \le \frac{2}{e^{(L-y)\sqrt{\beta}}}\|u(L)\|. \qquad (5)$$

Proof. From relations (1'') and (2) we have

$$u(y)-u_\beta(t)=\int_\beta^{+\infty} cosh\left(y\sqrt{\lambda}\right)dE_\lambda f \qquad (6)$$

$$=\int_\gamma^{+\infty} cosh\left(y\sqrt{\lambda}\right)\mathbf{1}_{[\boldsymbol{\beta},+\infty]}dE_{\boldsymbol{\lambda}}f$$

Then

$$u(y)-u_\beta(y)$$

$$=\int_\lambda^{+\infty}\frac{cosh\left(y\sqrt{\lambda}\right)}{cosh\left(L\sqrt{\lambda}\right)}1_{[\beta,+\infty]}cosh\left(L\sqrt{\lambda}\right)dE_\lambda f,$$

$$\left\|u(y)-u_\beta(y)\right\|^2$$

$$\le\int_\lambda^{+\infty}\left(\frac{cosh\left(y\sqrt{\lambda}\right)}{cosh\left(L\sqrt{\lambda}\right)}1_{[\beta,+\infty]}\right)^2 cosh\left(L\sqrt{\lambda}\right)^2 d\|E_\lambda f\|^2. \qquad (7)$$

Using the inequality

$$\left(\frac{cosh\left(y\sqrt{\lambda}\right)}{cosh\left(L\sqrt{\lambda}\right)}1_{[\beta,+\infty]}\right)^2 \le \frac{4}{e^{2(L-y)\sqrt{\beta,}}}$$

$$\int_\beta^{+\infty} cosh^2\left(L\sqrt{\lambda}\right)d\|E_\lambda f\|^2 \le \|u(L)\|^2, \qquad (8)$$

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We derive

$$\|u(y)\|^2 \le \frac{4}{e^{2(L-y)\sqrt{\beta}}}\|u(L)\|^2. \qquad (9)$$

Using (4), (5) and the triangle inequality, we obtain

$$\|u(y) - v_\beta^\delta(y)\|$$

$$\le \|u(y) - u_\beta(y)\| + \|u_\beta(y) - v_\beta^\delta(y)\| \qquad (10)$$

$$\le \frac{2}{e^{(L-y)\sqrt{\beta}}}\|u(L)\| + e^{y\sqrt{\beta}}\delta.$$

This completes the proof. □

Remark 1: *If we choose* $\sqrt{\beta} = (1/L)log(M/\delta)$, *where* $\|u(L)\| = M$, *then we have the error bound*

$$\|u(y) - v_\beta^\delta(y)\| \le 3M^{(L-y)/L}\delta^{y/L}. \qquad (11)$$

From (11) we see that (3) is an approximation of the exact solution $u(y)$. *The approximation error depends continuously on the measurement error for fixed* $0 < y < L$.

However, as $y \to L$, *the accuracy of the regularized solution becomes progressively lower. Consequently, we have note any information about the continuous dependence of the solution if* y *is close to* L.

In the theory of ill-posed Cauchy problems, we can often obtain continuous dependence on the data for the closed interval $[0, L]$ *by assuming additional smoothness and using a stronger norm.*

Now we show two error estimates under the following conditions:

$$(H1) u(L) \in D(A^p),$$

$$(H2) u(L) \in G_p, p > 0.$$

Remark 2: *In practice, we know that it is very difficult to verify the conditions* $(H1)$ *and* $(H2)$, *so we give different assumptions on the given data* f *as follows:*

$$u(L) \in D(A^p) \Leftrightarrow \int_\gamma^{+\infty} \lambda^{2p} cosh^2(L\sqrt{\lambda})d\|E_\lambda f\|^2 < \infty$$

$$\Leftrightarrow \int_{\gamma}^{+\infty} \lambda^{2p} e^{2L\sqrt{\lambda}} d\|E_\lambda f\|^2 < \infty,$$

$$u(L) \in G_p \Leftrightarrow \int_{\gamma}^{+\infty} \lambda^{2pL\sqrt{\lambda}} cosh^2\big(L\sqrt{\lambda}\big) d\|E_\lambda f\|^2 < \infty$$

$$\Leftrightarrow \int_{\gamma}^{+\infty} e^{2(1+p)L\sqrt{\lambda}} d\|E_\lambda f\|^2 < \infty,$$

$$\Leftrightarrow f \in G_{p+1.} \qquad (12)$$

Theorem 3: *If* $\int_{\gamma}^{+\infty} \lambda^{p} e^{2L\sqrt{\lambda}} d\|E_\lambda f\|^2 < E_1^2 \ \Big(resp., \int_{\gamma}^{+\infty} e^{2(1+q)L\sqrt{\lambda}} d\|E_\lambda f\|^2 < E_2^2,\Big) p > 0, q > 0, then\ one\ has\ the\ following\ estimates:$

$$\big\|u(y) - v_\beta^\delta(y)\big\|$$

$$\leq \left(\frac{L}{a}\right)^p E_1 log\left(\frac{1}{\delta}\right)^{-p} + \delta^{1-\frac{ya}{L},}\ \ 0 < a \leq 1, \qquad (13)$$

$$\big\|u(y) - v_\beta(y)\big\| \leq e^{-q\sqrt{\beta}} E_2 + e^{y\sqrt{\beta}}\delta.$$

Proof. From the expansions

$$u(y) = \int_{\gamma}^{+\infty} cosh\big(y\sqrt{\lambda}\big) dE_\lambda f,$$

$$u_\beta(y) = \int_{\gamma}^{+\infty} cosh\big(y\sqrt{\lambda}\big) \mathbf{1}_{[\gamma,\beta]} dE_\lambda f, \qquad (14)$$

We have

$$u(y) - u_\beta(y) = \int_{\gamma}^{+\infty} cosh\big(y\sqrt{\lambda}\big) 1_{[\beta,+\infty]} dE_\lambda f. \qquad (15)$$

Then

$$\big\|u(y) - u_\beta(u)\big\|^2$$

$$= \int_{\gamma}^{+\infty} \big(\lambda^{-p/2} \mathbf{1}_{[\beta,+\infty]}\big)^2 cosh^2\big(y\sqrt{\lambda}\big) \lambda^p d\|E_\lambda f\|^2$$

$$\leq \int_{\gamma}^{+\infty} \left(\lambda^{-p/2}\mathbf{1}_{[\beta,+\infty]}\right)^2 cosh^{2e^{2L\sqrt{\lambda}}\lambda^p} d\|E_\lambda f\|^2 \qquad (16) \qquad \leq$$

$$\beta^{-p} \int_{\gamma}^{+\infty} e^{2L\sqrt{\lambda}}\lambda^p d\|E_\lambda f\|^2$$

$$\leq \sqrt{\beta}^{-2p} E_1^2.$$

Using theorem 3 and the triangle inequality, we can write

$$\left\|u(y) - v_\beta^\delta(y)\right\|$$

$$\leq \left\|u(y) - u_\beta(y)\right\| + \left\|u_\beta(y) - v_\beta^\delta(y)\right\| \qquad (17)$$

$$\leq \sqrt{\beta}^{-p} E_1 + e^{y\sqrt{\beta}}\delta.$$

By choosing, $\sqrt{\beta} = (a/L)\ log(1/\delta)$*, we obtain the desired inequality.*

Using the same techniques, we have

$$u(y) - u_\beta(y)$$

$$= \int_{\gamma}^{\infty} e^{-q\sqrt{\lambda}} cosh\left(y\sqrt{\lambda}\right) e^{q\sqrt{\lambda}} \mathbf{1}_{[\beta,+\infty]} dE_\lambda f, \qquad (18)$$

Hence

$$\left\|u(y) - u_\beta(y)\right\|^2$$

$$= \int_{\gamma}^{\infty} \left(e^{-q\sqrt{\lambda}}\mathbf{1}_{[\beta,+\infty]}\right)^2 \left(cosh\left(y\sqrt{\lambda}\right)e^{2q\sqrt{\lambda}}\right)^2 d\|E_\lambda f\|^2$$

$$\leq e^{-2q\sqrt{\lambda}} d\|E_\lambda u(y)\|^2 \leq e^{-2q\sqrt{\beta}} E_2^2 \qquad (19)$$

Using (4) and the triangle inequality, we obtain

$$\left\|u(y) - v_\beta^\delta(y)\right\|$$

$$\leq \left\|u(y) - u_\beta(y)\right\| + \left\|u_\beta(y) - v_\beta^\delta(y)\right\| \quad (20)$$

$$\leq e^{-q\sqrt{\beta}}E_2 + e^{y\sqrt{\beta}}\delta.$$

By choosing, $\sqrt{\beta} = (a/L)\, log(1/\delta)$*, we obtain*

$$\left\|u(y) - v_\beta^\delta(y)\right\| \leq \delta^{aq/L}E_2 + \delta^{1-ay/L} \qquad (21)$$

2.2. The Mollification Method

Now, we approximate the original problem (1) by the sequence of problems

$$u_{yy} = Au, \qquad 0 < y < L,$$

$$u(0)=f_\propto = M_\propto f, \qquad (22)$$

$$u_y(0) = 0.$$

Theorem 4: *If* $f \in H$ *the approximate Cauchy problem (22) admits a unique solution* $u_{\propto,}$ *which depends continuously upon the data* f *with respect to uniform topology of* $C([0, L]; H)$.

Proof. From the representation

$$u_\propto(y) = cosh\big(y\sqrt{A}\big)f_\propto$$

$$= \int_\gamma^{+\infty} cosh\big(y\sqrt{\lambda}\big)\left(1 + \alpha e^{pL\sqrt{\lambda}}\right)^{-1} dE_\lambda f, \qquad (23)$$

We have

$$\|u_\alpha(y)\|^2 = \int_\gamma^{+\infty} \left\{\frac{cosh\big(y\sqrt{\lambda}\big)}{1+\alpha e^{pL\sqrt{\lambda}}}\right\}^2 d\|E_\lambda f\|^2$$

$$\leq \int_\gamma^{+\infty} \left\{\frac{e^{L\sqrt{\lambda}}}{1+\alpha e^{pL\sqrt{\lambda}}}\right\}^2 d\|E_\lambda f\|^2. \qquad (24)$$

[1]. If $p = 1$, we obtain

$$\sup_{y\epsilon[0,L]} \|u_\alpha(y)\|^2 \leq \frac{1}{\alpha}\|f\|. \qquad (25)$$

[2]. *If* $p > 1$, *the function* $M(s) = e^{Ls}/(1 + \alpha e^{pLs})$ *with* $s = \sqrt{\lambda} \geq \sqrt{\gamma}$ *achieves its maximum at* $s^* =$

$(1/pL)\ log\big(1/\alpha(p-1)\big), p > 1,$ *from which we deduce*

$$M_\infty = M(S^*) = c(p)\left(\frac{1}{\alpha}\right)^{1/p}, \qquad (26)$$

$$c(p) = p^{-1}(p-1)^{1-1/p} \le 1.$$

From this bound, we drive

$$\sup_{y\in[0,L]} \|u_\alpha(y)\| \le \left(\frac{1}{\alpha}\right)^{\frac{1}{p}} \|f\|. \qquad (27)$$

From the linear property of our problem, stability estimate of problem (22) may be written precisely in the following corollary. □

Corollary 1: *If* $u_{\alpha,1}(y;f_1)\ \big(resp,.\, u_{\alpha,2}(y,f_2)\big)$ *is the approximate solution corresponding to* $f_1(resp,.\ f_2)$*, then*

$$\sup_{y\in[0,L]} \left\|u_{\alpha,1}(y) - u_{\alpha,2}(y)\right\| \le \left(\frac{1}{\alpha}\right)^{\frac{1}{p}} \|f_1 - f_2\| \qquad (28)$$

Remark 3: *We have*

$$N(s) = \frac{s^r e^{Ts}}{1+\alpha e^{pTs}} \le \frac{1}{\alpha} k(s)$$

$$= \frac{1}{\alpha} s^r e^{-(p-1)Ts}, \qquad p > 1, \qquad (29)$$

$$s = \sqrt{\lambda} \ge \sqrt{\gamma}.$$

It is easy to show that

$$K(s) \le K\left(s^* = \frac{r}{L(p-1)}\right) \qquad (30)$$

$$= \left(\frac{r}{L(p-1)}\right)^r e^{-r} = k(r,p,L) < \infty.$$

This remark shows that $u_\alpha(y) \in D\big(A^{r/2}\big)$ *for all* $y \in [0,L]$.

Proof. The inclusion $u_\alpha(y)\ \in\ D\big(A^{r/2}\big)$ *is equivalent to* $\left\|A^{r/2}u_\alpha(y)\right\| < \infty$. *We have*

$$\left\|A^{r/2}u_\alpha(y)\right\|^2 = \int_\gamma^{+\infty} \lambda^r \left\{\frac{cosh\left(y\sqrt{\lambda}\right)}{1+\alpha e^{pL\sqrt{\lambda}}}\right\}^2 d\|E_\lambda f\|^2$$

$$\leq \left(\frac{1}{\propto}\right)^2 \int_\gamma^{+\infty} \left\{\sqrt{\lambda}^r e^{-(p-1)L\sqrt{\lambda}}\right\}^2 d\|E_\lambda f\|^2$$

$$\leq \left(\frac{1}{\propto}\right)^2 k(r,p,L)^2\|f\|^2 < \infty, \qquad (31)$$

Where $k(r,p,L) = sup_{\lambda\geq\gamma}\sqrt{\lambda}^r e^{-(p-1)L\sqrt{\lambda}} = \left(\frac{r}{(p-1)L}\right)^r e^{-r}$. □

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Theorem 5: *If* $f \in G_1$, *then*

$$\sup_{y\in[0,L]}\|u(y)-u_\alpha(y)\| \to 0, \;\; \alpha \to 0. \qquad (32)$$

Proof. We compute

$$\|u(y)-u_\alpha(y)\|^2$$

$$= \int_\gamma^{+\infty} \left(1-M_\alpha(\lambda)\right)^2 cosh^2\left(y\sqrt{\lambda}\right) d\|E_\lambda f\|^2$$

$$\leq \int_\gamma^{+\infty} \left(1-M_\alpha(\lambda)\right)^2 cosh^2\left(L\sqrt{\lambda}\right) d\|E_\lambda f\|^2$$

$$\leq \int_\gamma^{+\infty} \left(1-M_\alpha(\lambda)\right)^2 e^{2L\sqrt{\lambda}}\left(y\sqrt{\lambda}\right) d\|E_\lambda f\|^2$$

$$= \left\|(I-M_\alpha)\hat{f}\right\|^2,$$

Where $\hat{f} = e^{L\sqrt{\lambda}}f$ *and* $\left\|\hat{f}\right\|^2 = \int_\gamma^{+\infty} e^{2L\sqrt{\lambda}} d\|E_\lambda f\|^2 < \infty$.

This implies that $\sup_{y\in[0,L]}\|u(y)-u_\alpha(y)\| \leq \left\|(I-M_\alpha)\hat{f}\right\|$

And by virtue of (1) of theorem , we conclude the desired convergence.

The following technical lemmas play the key role in our analysis and calculations.

Lemma 1: *Let*

$$[v, +\infty[\ \ni s \to Q(\{a, r, q, L\}; s) = \frac{1}{\alpha s^r + a e^{-qLs}}\,, \qquad (34)$$

Where $a > 0, \alpha > 0, v > 0,\ q > 0,\ L > 0,$ *and* $r \geq 1$. *Then one has*

$$Q(\{a, r, q, L\}; s) \leq \frac{1}{\alpha}\left(\frac{k_1}{log(k_2(1/\alpha))}\right)^r, \qquad (35)$$

Where $k_1(r, q, L) = rqL, k_2(q, r, L, a) = q^r L^{r-1} a/r$.

Proof. Differentiating the expression and setting the derivative equal to zero, we find

$$\frac{d}{ds} Q(\{a, r, q, L\}; s)$$

$$= \frac{-1}{(\alpha s^r + a e^{-qLs})^2}(\alpha r s^{r-1} - q a e^{-qLs}) = 0. \qquad (36)$$

The function $(d/ds)\ Q(\{a, r, q, L\}; s) = 0$ *admits a unique solution*

$$s = \{s \longmapsto \alpha r s^{r-1}\} \cap \{s \longmapsto q a e^{-qLs}\}. \qquad (37)$$

Therefore

$$Q(\{a, r, q, L\}; s) \leq Q(\{a, r, q, L\}; \hat{s})$$

$$\leq \frac{1}{\alpha \hat{s}^r + a e^{-qLs}} \leq \frac{1}{\alpha \hat{s}^r}. \qquad (38)$$

We have

$$\left(\alpha r \hat{s}^{r-1} - q a e^{-qL\hat{s}}\right) = 0 \Leftrightarrow s^{r-1} e^{qL\hat{s}} = \frac{q^a}{r\alpha}. \qquad (39)$$

By using the inequality $(e^t \geq t, t \geq 0)$, *then for* $t = qL\widehat{s}$, *we obtain* $e^{qL\hat{s}} \geq qL\widehat{s}$ *and we can write*

$$\frac{q^a}{r\alpha} = \hat{s}^{r-1} e^{qL\hat{s}} \leq e^{qL\hat{s}} \left(\frac{e^{qL\hat{s}}}{qL}\right)^{r-1} = \left(\frac{1}{qL}\right)^{r-1} e^{rqL\hat{s}}, \qquad (40)$$

Which implies that $\geq \left(\frac{1}{rqL}\right)\ log\left(\left(\frac{q^r L^{r-1} a}{r}\right)\left(\frac{1}{\alpha}\right)\right)$. *Hense, we obtain*

$$Q(\{a, r, q, L\}; s) \leq \frac{1}{\alpha \hat{s}^r} \leq \frac{1}{\alpha}\left(\frac{k_1}{log\left(k_2\left(\frac{1}{\alpha}\right)\right)}\right)^r, \qquad (41)$$

$Where\ k_1(r,q,L) = rqL, k_2(q,r,L,a) = q^r L^{r-1} a/r.$ □

Lemma 2: *Let*

$$[v,+\infty[\ \ni s \mapsto R(\{p,q,L\};s)$$

$$= \frac{e^{pLs}}{(1+\alpha e^{pLs})e^{qLS}} = \frac{1}{e^{(q-p)Ls} + \alpha e^{qLs}}, \qquad (42)$$

$Where\ p \geq 1, q > 0,\ \alpha > 0, v > 0, and\ L > 0. Then\ on\ has\ the\ following.$

$If\ 1 \leq p \leq q, then$

$$R(\{p,q,L\};s) \leq e^{-(q-p)Ls} \leq e^{-(q-p)Lv} \leq 1. \qquad (43)$$

$If\ 0 < q < p, p \geq 1,\ 0 < \propto \leq (p-q)/q, then$

$$R(\{p,q,L\};s) \leq k_3 \left(\frac{1}{\propto}\right)^{(p-q)/p}$$

$$k_3(p,q) = \frac{q}{p}\left(\frac{p-q}{p}\right)^{(p-q)/p} \leq 1. \qquad (44)$$

$Proof. By\ a\ simple\ differrential\ calculus, we\ show\ that\ the\ function\ R(\{p,q,L\};s)$

$achieves\ its\ maximum\ at\ \hat{s} = (1/pL)\ log\big((p-q)/\propto q\big). Consequently$

$$R(\{p,q,L\};s) \leq R(\{p,q,l\};\hat{s}) = k_3 \left(\frac{1}{\propto}\right)^{\frac{(p-q)}{p}}. \qquad (45)$$

Now we assume the following a priori bounds hold: □

$u(L) \in D\big(A^{r/2}\big)$

$$\Leftrightarrow \int_{\gamma}^{+\infty} \sqrt{\lambda}^{2r} e^{2L\sqrt{\lambda}}\, d\|E_\lambda f\|^2 \leq E_1^2 < \infty, \qquad (46)$$

$u(L) \in G_q$

$$\Leftrightarrow \int_{\gamma}^{+\infty} \sqrt{\lambda}^{2r} e^{2(1+q)\sqrt{\lambda}}\, d\|E_\lambda f\|^2 \leq E_1^2 < \infty. \qquad (47)$$

Theorem 6: $Let\ u(resp., u_\alpha)\ be\ the\ solution\ of\ problem\ (1'')\ \big(resp.,\ (22)\big)$

With the exact data $f.\ If\ (46)\ \big(resp., (47)\big)\ is\ satisfied, then\ \ on\ has\ the$

Following error estimates:

$$\|u(y) - u_{\propto}(y)\| = 0\left(\frac{1}{log(1/\alpha)}\right)^{r}, \qquad (48)$$

$\|u(y) - u_{\alpha}(y)\|$

$$= \begin{cases} 0(\alpha), & if\ 1 \leq p \leq q, \\ 0\big(\alpha^{q/p}\big), & if\ 0 < q < p, p \geq 1. \end{cases} \qquad (49)$$

Proof. Putting

$$B_1(\lambda) = \left\{\frac{e^{pL\sqrt{\lambda}}}{1 + \alpha e^{pL\sqrt{\lambda}}}\right\} \frac{1}{\sqrt{\lambda}^{r}}$$

$$= \frac{1}{\sqrt{\lambda}^{r} e^{-pL\sqrt{\lambda}} + \alpha\sqrt{\lambda}^{r}} \leq B_2(\lambda)$$

$$= \frac{1}{\sqrt{\lambda}^{r} e^{-pL\sqrt{\lambda}} + \alpha\sqrt{\lambda}^{r}},$$

$$B_3(\lambda) = \left\{\frac{e^{pL\sqrt{\lambda}}}{1 + \alpha e^{pL\sqrt{\lambda}}}\right\} \frac{1}{e^{qL\sqrt{\lambda}}} = \frac{1}{e^{(q-p)L\sqrt{\lambda}} + \alpha e^{qL\sqrt{\lambda}.}}$$

Using the change of variable, $s = \sqrt{\lambda}$, *we obtain the new expressions*

$$\hat{B}_2(s) = \frac{1}{\sqrt{\gamma}^{r} e^{-pLs} + \alpha e^{r}}, \qquad (51)$$

$$\hat{B}_3(s) = \frac{1}{e^{(q-p)Ls} + \alpha e^{qLs.}} \qquad (52)$$

By virtue of lemma $1\ (inequality\ (35) and\ Lemma\ 2\ (inequalities\ (43)$

And $(69)), we\ can\ write$

$$\hat{B}_2(s) \leq \frac{1}{\alpha}\left(\frac{k_1}{log\left(k_2\left(\frac{1}{\alpha}\right)\right)}\right)^{r}, \qquad (53)$$

Where $k_1(r, p, L) = rqL, k_2\big(p, r, L, \sqrt{\gamma}^{r}\big) = q^{r}, L^{r-1}\sqrt{\gamma}^{r/r.}$ *Consider*

$$\hat{B}_3(s) \leq \begin{cases} 1, & if\ 1 \leq p \leq q, \\ \left(\frac{1}{\alpha}\right)^{(p-q)/p,}, & if o < q < p, p \geq 1. \end{cases} \qquad (54)$$

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We Have

$\|u(y) - u_\alpha(y)\|^2$

$$= \int_\gamma^{+\infty} \left\{\frac{\alpha e^{pL\sqrt{\lambda}}}{1+\alpha e^{pL\sqrt{\lambda}}}\right\}^2 cosh^2\left(y\sqrt{\lambda}\right) d\|E_\lambda f\|^2$$

$$\leq \alpha^2 \int_\gamma^{+\infty} \{B_2(\lambda)\}^2 \sqrt{\lambda}^{-2r} e^{2L\sqrt{\lambda}}\ d\|E_\lambda f\|^2$$

$$\leq \alpha^2 \begin{pmatrix} sup\ \hat{B}_2(s) \\ s \geq \sqrt{\gamma} \end{pmatrix}^2 E_1^2, \qquad (55)$$

$$\|u(y) - u_\alpha(y)\|^2$$

$$= \int_\gamma^{+\infty} \left\{\frac{\alpha e^{pL\sqrt{\lambda}}}{1+\alpha e^{pL\sqrt{\lambda}}}\right\}^2 cosh^2\left(y\sqrt{\lambda}\right) d\|E_\lambda f\|^2$$

$$\leq \alpha^2 \int_\gamma^{+\infty} \{B_3(\lambda)\}^2 e^{L(1+q)\sqrt{\lambda}}\ d\|E_\lambda f\|^2$$

$$\leq \alpha^2 \begin{pmatrix} sup\ \hat{B}_2(s) \\ s \geq \sqrt{\gamma} \end{pmatrix}^2 E_2^2.$$

Using (53) and (54), we drive

$$\|u(y) - u_\alpha(y)\|^2 \leq \alpha \frac{1}{\alpha}\left(\frac{k_1}{log\left(k_2\left(\frac{1}{\alpha}\right)\right)}\right)^r = 0\left(\frac{1}{log(k_2(1/\alpha))}\right)^r,$$

$$\|u(y) - u_\alpha(y)\|^2 \leq \begin{cases} \alpha, & if\ 1 \leq p \leq q, \\ \alpha^{q/p}, & if\ 0 < q < p, p \geq 1. \end{cases} \qquad (56)$$

Combining(28), *(48), and (49) with the help of triangle inequality* □

$\|u(y) - u_\alpha^\delta(y)\|$

$$\leq \|u(y) - u_\alpha(y)\| \qquad (57)$$

$+\|u(y) - u_\alpha^\delta(y)\| = \Delta_1 + \Delta_{2,}$

We deduce the following corollary.

Corollary 3: *Let* $u(y; f)(resp., u_\alpha^\delta(y; f_\delta))$ *be the solution of problem (1) (resp., (47)) with the exact data* f *(resp., the inexact data* f_δ*) such that* $\|f - f_\delta\| \leq \delta. If\ (46)(resp., (47))\ is\ satisfied, then\ one\ has\ the\ following\ error$ *estimates:*

$(case\ r \geq 1)$

$$\|u(y) - u_\alpha^\delta(y)\| = 0(\theta_1(\alpha)) + \left(\frac{1}{\alpha}\right)^{1/p} \delta, \qquad (58)$$

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(Case $1 \leq p \leq q$*)*

$$\|u(y) - u_\alpha^\delta(y)\| = 0(\theta_1(\alpha)) + \left(\frac{1}{\alpha}\right)^{1/p} \delta, \qquad (59)$$

(Case 0< $q < p, p \geq 1$*)*

$$\|u(y) - u_\alpha^\delta(y)\| = 0(\theta_3(\alpha)) + \left(\frac{1}{\alpha}\right)^{1/p} \delta, \qquad (60)$$

Where

$$\theta_1(\alpha) = 0\left(\frac{1}{log(1/\alpha)}\right)^r, \qquad (61)$$

$\theta_2(\alpha) = 0(\alpha), \qquad \theta_2(\alpha) = 0(\alpha^{q/p}).$

If we choose $\alpha = \alpha(\delta) = \delta^{p/\omega}$ $with\ \omega > 1, then\ we\ have$

$$\delta\left(\frac{1}{\delta^{p/\omega}}\right)^{1/p} = \delta^{(\omega-1)/\omega,} \qquad (62)$$

$$\theta_1(\alpha) = 0\left(\frac{1}{log(1/\delta^{p/\omega})}\right)^r, \qquad (63)$$

$$\theta_2(\alpha) = \delta^{p/\omega}, \quad \theta_3(\alpha) = 0(\alpha^{q/\omega}). \qquad (64)$$

2.3. Example: Cauchy Problem for the Modified Helmholtz equation

In this paragraph, we give a concrete example to see how to apply the theoretical results developed in this Study.

Let us consider the Cauchy problem (modified Helmholtz equation) in the infinite strip ℝ x (0, 1)

$$u_{yy}(x,y) + u_{xx}(x,y) - yu(x,y) = 0, x \in \mathbb{R}, \quad y \in (0,1), \qquad (65)$$

$$u(x,0) = f(x), u_y(x,0) = 0, \quad x \in \mathbb{R},$$

Where y is a real positive constant.

Let $\hat{u}(\xi,y) = (\mathfrak{F}u)(\xi,y)$ *be the Fourier transform of* $u(x,y)$:

$$\hat{u}(\xi,y) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e\xi^x u(x,y)dx. \qquad (66)$$

With the help of the Fourier transformed, problem (1'') can be transformed to an equivalent problem in the frequency domain:

$$\hat{u}_{yy}(\xi,y) - \xi^2\hat{u}(\xi,y) - y\hat{u}(\xi,y) = 0,$$

$$\xi \in \mathbb{R}, \qquad y \in (0,1), \qquad (67)$$

$$\hat{u}(\xi,0) = \hat{f}(\xi), \hat{u}_y(\xi,0) = 0, \quad \xi \in \mathbb{R}.$$

It is easy to check that the formal solution of problem (67) has the form

$$\hat{u}(\xi,y) = cosh\left(y\sqrt{(\xi^2+\gamma)}\right)\hat{f}(\xi), (68)$$

Or equivalently, the formal solution of problem (65) is given by

$$u(x,y) = (\mathfrak{F}^{-1}\hat{u})(x,y)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\hat{u}(\xi,y)d\xi \qquad (69)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}cosh\left(y\sqrt{(\xi^2+\gamma)}\right)\hat{f}(\xi)d\xi.$$

Putting $\Theta(\xi) = \sqrt{(\xi^2+\gamma)}$*. Then* $\Theta(\xi) \longrightarrow +\infty$ $|\xi| \longrightarrow +\infty$ *From this remark, it is easy to see that a small perturbation in data* $\hat{f}(\xi)$ *may cause a dramatically large error in the solution* $\hat{u}(\xi,\xi)$*. In addition, the magnifying factor is* $\Theta(\xi) \sim e^{|\xi|}$*, hence, the problem is severely ill-posed.*

Since the data $f(.)$ *are based on (physical) observations and are not known with complete accuracy, we* *assume that* f *and* f_δ *satisfy*

$$\|f - f_\delta\| \leq \delta, \qquad (70)$$

Where f *and* f_δ *belong to* $L^2(\mathbb{R})$, f_δ *denotes the measured data, and* δ *denotes the noise level.*

For this problem, we define the regularized solutions with noisy data f_δ:

$u_N^\delta(x,y)$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} i^{x\xi} cosh\left(y\sqrt{(\xi^2+\gamma)}\right) \hat{f}_\delta(\xi) \mathbf{1}_{[-N,N]}(\xi) d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} e^{ix\xi} \, cosh\left(y\sqrt{(\xi^2+\gamma)}\right) \hat{f}_\delta(\xi) d\xi \qquad (71)$$

Where $1_{1[-N,N]}$ *is the characteristic function of the interval* $[-N, N]$

$u_\propto^\delta(x,y)$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \left(\frac{cosh\left(y\sqrt{(\xi^2+\gamma)}\right)}{1+\propto e^{p\sqrt{(\xi^2+\gamma)}}} \right) \hat{f}_\delta(\xi) d\xi$$

Where $p \geq 1$. *The quantities* $\propto = \propto(\delta)$ *and* $N = N(\delta)$ *are the parameters which were defined in Sections 3.1 and 3.2.*

3. Conclusion

We were able to solve the problem with the truncation method and the mollification method. Our goal will be devoted to problematic waters with unknown (uncertain) operators: Since the physical model proceeds from an idealization of physical reality and is based on simplifying assumptions, it is therefore also a source of uncertainty. Any regularization theory must therefore take into account the possibly incomplete for uncertain character. Also, we give some extensions to our investigation.

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